

Modeling the evolution of populations undergoing selection using branching Brownian motion

Lecture I

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Overview

Goal of population genetics: understand the factors, such as mutation and natural selection, that cause genetic variability.

Natural selection: individuals acquire beneficial mutations which increase fitness.

Questions of interest:

1. How fast does the fitness level of the population change over time?
2. What is the distribution of the fitness levels of individuals in the population at a given time?
3. How can we describe the genealogy of the population?

We will model populations undergoing selection using branching Brownian motion.

Outline of the lecture

This lecture will focus on background material involving population models, branching, and coalescence:

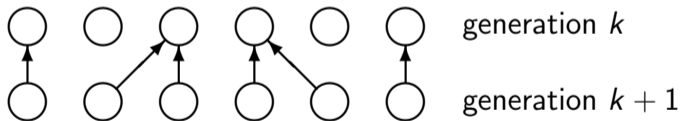
1. Classical population models and Kingman's coalescent
2. Coalescents with multiple mergers
3. Genealogy of continuous-state branching processes
4. Two population models involving selection

In the remaining four lectures, we will analyze in depth approximations to the two population models involving branching Brownian motion.

The Wright-Fisher Model

One of the earliest models in population genetics, goes back to Fisher (1921) and Wright (1930).

- ▶ The population has fixed size $2N$.
- ▶ Generations do not overlap.
- ▶ Each member of the population has one parent, chosen at random from the individuals in the previous generation.

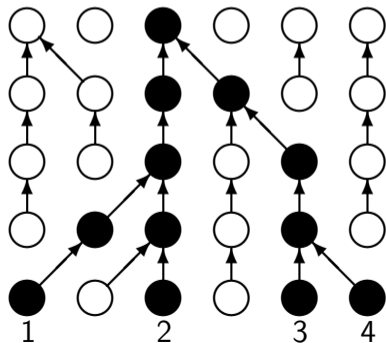


Members of population can be viewed as individuals in a haploid population or as chromosomes in a diploid population.

Ancestral Processes

Sample n individuals at the present time (generation 0).

Let $\Psi_N(k)$ be the partition of $\{1, \dots, n\}$ such that i and j are in the same block if and only if the i th and j th sampled individuals have the same ancestor in generation $-k$.



$$\Psi_N(4) = \{\{1, 2, 3, 4\}\}$$

$$\Psi_N(3) = \{\{1, 2\}, \{3, 4\}\}$$

$$\Psi_N(2) = \{\{1, 2\}, \{3, 4\}\}$$

$$\Psi_N(1) = \{\{1\}, \{2\}, \{3, 4\}\}$$

$$\Psi_N(0) = \{\{1\}, \{2\}, \{3\}, \{4\}\}$$

Merging of two ancestral lines

Consider two individuals in generation 0. The probability that they have the same parent is $1/2N$.

Let T be the number of generations we have to go back before they have the same ancestor. Then

$$P(T > k) = \left(1 - \frac{1}{2N}\right)^k.$$

In particular,

$$P(T > 2Nx) = \left(1 - \frac{1}{2N}\right)^{\lfloor 2Nx \rfloor} \approx e^{-x}.$$

$T/2N$ has approximately an exponential distribution with rate 1.

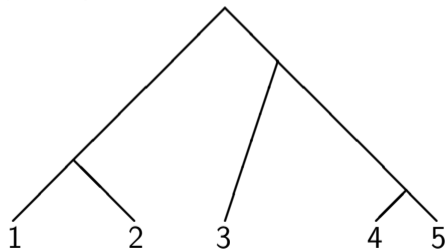
The probability that three individuals in some generation all have the same parent is $1/(2N)^2$, so it is unlikely that three or more ancestral lines will merge simultaneously.

Kingman's Coalescent (Kingman, 1982)

Continuous-time Markov chain $(\Pi_n(t), t \geq 0)$ taking values in the set \mathcal{P}_n of partitions of $\{1, \dots, n\}$.

Here $\Pi_n(0)$ consists of n singletons. Any transition that involves merging two blocks into one happens at rate 1. No other transitions are possible.

When there are k blocks, the distribution of the time until the next merger is exponential with rate $k(k-1)/2$. Then two randomly chosen blocks merge.



One time unit in Kingman's n -coalescent corresponds to $2N$ generations in the Wright-Fisher model.

A Limit Theorem

Theorem (Kingman, 1982): Suppose a population evolves according to the Wright-Fisher model with population size $2N$. Sample n individuals at random from the population in generation zero. Let $\Psi_N(k)$ be the partition of $\{1, \dots, n\}$ such that i and j are in the same block if and only if the i th and j th sampled individuals have the same ancestor in generation $-k$. Let $(\Pi_n(t), t \geq 0)$ be Kingman's coalescent. Then, as $N \rightarrow \infty$,

$$(\Psi_N(\lfloor 2Nt \rfloor), t \geq 0) \Rightarrow (\Pi_n(t), t \geq 0).$$

Here \Rightarrow denotes weak convergence of stochastic processes with respect to the Skorohod topology.

The Moran Model

Continuous-time model introduced by Moran (1958).

- ▶ The population has fixed size $2N$.
- ▶ Each individual independently lives for an Exponential(1) time, then is replaced by a new individual.
- ▶ If a new individual is born at time t , its parent is chosen uniformly at random from the individuals alive at time $t-$.
- ▶ The population can be defined for all $t \in \mathbb{R}$.

Theorem: Suppose we sample n individuals at random from the population at time 0. Let $\Psi_N(t)$ be the partition of $\{1, \dots, n\}$ such that i and j are in the same block if and only if the i th and j th individuals in the sample have the same ancestor at time $-t$. Then

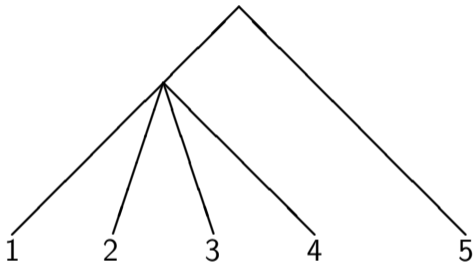
$$(\Psi_N(Nt), t \geq 0)$$

is Kingman's coalescent.

Merger rate of two lineages is $\binom{2}{2}(1/2N) = 1/N$.

Coalescents with multiple mergers (Λ -coalescents)

Introduced by Pitman (1999), Sagitov (1999), and Donnelly and Kurtz (1999).
More than two ancestral lines can merge at a time.



Applications of coalescents with multiple mergers:

- ▶ Large family sizes (many lineages trace back to an individual with large number of offspring).
- ▶ Natural selection (many lineages trace back to an individual who got a beneficial mutation).

Definition and construction of the Λ -coalescent

Let Λ be a finite measure on $[0, 1]$. The Λ -coalescent is a continuous-time \mathcal{P}_n -valued Markov chain such that when there are b blocks, each k -tuple ($2 \leq k \leq b$) of blocks merges at rate

$$\lambda_{b,k} = \int_0^1 p^{k-2}(1-p)^{b-k} \Lambda(dp).$$

Put $\Lambda = a\delta_0 + \Lambda_0$, with $\Lambda_0(\{0\}) = 0$.

- ▶ Each pair of lineages merges at rate a .
- ▶ Construct a Poisson point process on $[0, \infty) \times (0, 1]$ with intensity

$$dt \times p^{-2}\Lambda_0(dp).$$

If (t, p) is a point of this Poisson process, then a p -merger occurs at time t . Each lineage independently participates with probability p .

Heavy-tailed offspring distributions

Consider the following population model:

- ▶ The population has fixed size N .
- ▶ Generations do not overlap.
- ▶ Numbers of offspring ξ_1, \dots, ξ_N of the N individuals are i.i.d. positive integer-valued random variables.
- ▶ We have $P(\xi_i \geq k) \sim Ck^{-\alpha}$, where $\alpha \geq 1$.
- ▶ Obtain the next generation by sampling N offspring without replacement.

Sample n individuals in generation 0.

Let $\Psi_N(k)$ be the partition of $\{1, \dots, n\}$ such that i and j are in the same block if and only if the i th and j th sampled individuals have the same ancestor in generation $-k$.

Genealogy of the population

Theorem (Schweinsberg, 2003): As $N \rightarrow \infty$,

- ▶ If $\alpha \geq 2$, the processes $(\Psi_N(\lfloor t/c_N \rfloor), t \geq 0)$ converge to Kingman's coalescent. When $\alpha > 2$, we have $c_N \sim \sigma^2/N$, where σ^2 is the variance of the number of surviving offspring.
- ▶ If $1 < \alpha < 2$, the processes $(\Psi_N(\lfloor AN^{\alpha-1}t \rfloor), t \geq 0)$ converge, for some constant A , to the Λ -coalescent, where Λ is the Beta($2 - \alpha, \alpha$) distribution (beta coalescent).
- ▶ If $\alpha = 1$, the processes $(\Psi_N(\lfloor (\log N)t \rfloor), t \geq 0)$ converge to the Λ -coalescent, where Λ is uniform distribution on $(0, 1)$, called the Bolthausen-Sznitman (1998) coalescent.

Idea of the proof ($1 < \alpha < 2$)

Let $\mu = E[\xi_i]$ be the mean of the offspring distribution.

We get a p -merger with $p \geq x$ if

$$\frac{\xi}{\xi + N\mu} \geq x \quad \iff \quad \xi \geq \frac{x}{1-x} \cdot N\mu$$

The probability of such a family in a given generation is

$$NP\left(\xi \geq \frac{x}{1-x} \cdot N\mu\right) \sim NC\left(\frac{x}{1-x} \cdot N\mu\right)^{-\alpha}.$$

The rate of such mergers in the Beta($2 - \alpha, \alpha$)-coalescent is

$$\frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_x^1 p^{-1-\alpha}(1-p)^{\alpha-1} dp = \frac{1}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)} \left(\frac{x}{1-x}\right)^{-\alpha}.$$

Bienaymé-Galton-Watson Processes

Definition: Let $(p_k)_{k=0}^{\infty}$ be a sequence of nonnegative numbers such that $\sum_{k=0}^{\infty} p_k = 1$. Consider a population with the following properties:

- ▶ There is one individual in generation zero.
- ▶ An individual has k offspring with probability p_k .
- ▶ The numbers of offspring of different individuals are independent.

Let Z_n be the population size in generation n . Then $(Z_n)_{n=0}^{\infty}$ is a *Bienaymé-Galton-Watson process* with offspring distribution $(p_k)_{k=0}^{\infty}$.

Let L be a random variable such that $P(L = k) = p_k$ for $k = 0, 1, 2, \dots$.

Let $m = E[L]$. We say the process is *subcritical* if $m < 1$, *critical* if $m = 1$, and *supercritical* if $m > 1$.

Theorem: Let $q = P(Z_n = 0 \text{ for some } n)$ be the extinction probability. Then $q < 1$ if and only if $m > 1$ or $p_1 = 1$.

Limit theorems for Bienaymé-Galton-Watson Processes

Theorem: Suppose $m = 1$ and $\text{Var}(L) = \sigma^2 < \infty$. Then

- ▶ (Kolmogorov, 1938): As $n \rightarrow \infty$, we have

$$P(Z_n > 0) \sim \frac{2}{n\sigma^2}.$$

- ▶ (Yaglom, 1947): As $n \rightarrow \infty$, the conditional distribution of Z_n/n given $Z_n > 0$ converges to the exponential distribution with mean $\sigma^2/2$.

Theorem (Kesten and Stigum, 1966): Suppose $1 < m < \infty$. Then $E[Z_n] = m^n$ and

$$\lim_{n \rightarrow \infty} Z_n/m^n = W \quad \text{a.s.}$$

If $E[L \log^+ L] < \infty$, then $W > 0$ almost surely on nonextinction.

Theorem (Seneta, 1968; Heyde, 1970): If $1 < m < \infty$, then there exist constants $(c_n)_{n=0}^\infty$ such that almost surely on nonextinction,

$$\lim_{n \rightarrow \infty} Z_n/c_n \in (0, \infty).$$

Lévy Processes

Definition: $(X(t), t \geq 0)$ is called a *Lévy process* if:

- ▶ If $0 = t_0 < t_1 < \dots < t_n$, then the increments

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$$

are independent.

- ▶ If $s, t \geq 0$, then $X(t + s) - X(t)$ has the same distribution as $X(s) - X(0)$.
- ▶ Almost surely $t \rightarrow X(t)$ is right continuous.

Examples:

1. Brownian motion. Then $E[e^{iuX(t)}] = \exp(-\frac{1}{2}u^2t)$.
2. Deterministic drift at rate d . Then $E[e^{iuX(t)}] = \exp(idut)$.
3. Jumps of size x at times of a rate λ Poisson process. Then $E[e^{iuX(t)}] = \exp(\lambda t(e^{iux} - 1))$.

Lévy-Khintchine Formula

Suppose $(X(t), t \geq 0)$ is a Lévy process with $X(0) = 0$. There is a function Φ called the *characteristic exponent* such that $E[e^{iuX(t)}] = e^{t\Phi(u)}$. We have

$$\Phi(u) = idu - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux\mathbb{1}_{\{|x| \leq 1\}}) \nu(dx),$$

where $d \in \mathbb{R}$, $\sigma^2 \geq 0$, and ν is a Lévy measure with $\nu(\{0\}) = 0$ and $\int_{-\infty}^{\infty} (1 \wedge x^2) \nu(dx) < \infty$.

If the Lévy process has no negative jumps, then for $\lambda \geq 0$, we have $E[e^{-\lambda X(t)}] = e^{t\Psi(\lambda)}$, where

$$\Psi(\lambda) = -d\lambda + \frac{\sigma^2 \lambda^2}{2} + \int_0^{\infty} (e^{-\lambda x} - 1 + \lambda x\mathbb{1}_{\{x \leq 1\}}) \nu(dx).$$

The function $-\Psi$ is called the *Laplace exponent*.

Continuous-State Branching Processes

Definition: A continuous-state branching process (CSBP) is a $[0, \infty]$ -valued Markov process $(X(t), t \geq 0)$ whose transition functions satisfy

$$p_t(a + b, \cdot) = p_t(a, \cdot) * p_t(b, \cdot).$$

Theorem (Lamperti, 1967): CSBPs are the processes that can be obtained as limits of processes $(X_n(t), t \geq 0)$ with

$$X_n(t) = \frac{Z_n(\lfloor nt \rfloor)}{a_n},$$

where each Z_n is a Bienaymé-Galton-Watson process and $Z_n(0) \rightarrow \infty$ as $n \rightarrow \infty$.

CSBPs and Lévy Processes

Theorem (Lamperti, 1967): Let $(Y(s), s \geq 0)$ be a Lévy process with no negative jumps with $Y(0) > 0$, stopped when it hits zero. Let

$$S(t) = \inf\left\{u : \int_0^u Y(s)^{-1} ds > t\right\}.$$

Let $X(t) = Y(S(t))$. Then $(X(t), t \geq 0)$ is a CSBP. Every CSBP with no instantaneous jump to ∞ can be obtained this way.

Suppose $(X(t), t \geq 0)$ is a CSBP obtained from the Lévy process $(Y(t), t \geq 0)$. If $Y(0) = a$, then $E[e^{-\lambda Y(t)}] = e^{-\lambda at} e^{t\Psi(\lambda)}$, where

$$\Psi(\lambda) = -d\lambda + \frac{\sigma^2 \lambda^2}{2} + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x \mathbb{1}_{\{x \leq 1\}}) \nu(dx).$$

The function Ψ is called the *branching mechanism* of the CSBP.

Examples of CSBPs

1. Feller's branching diffusion: $\Psi(\lambda) = \frac{1}{2}\lambda^2$. Then $(X(t), t \geq 0)$ satisfies the SDE

$$dX(t) = \sqrt{X(t)} dB(t),$$

where $(B(t), t \geq 0)$ is standard Brownian motion.

2. α -stable CSBP with $1 < \alpha < 2$:

$$\Psi(\lambda) = \lambda^\alpha, \quad \nu(dx) = \frac{\alpha(\alpha - 1)}{\Gamma(2 - \alpha)} x^{-1-\alpha} dx.$$

3. Neveu's CSBP (Neveu, 1992):

$$\Psi(\lambda) = \lambda \log \lambda, \quad \nu(dx) = x^{-2} dx.$$

Defining the genealogy of a CSBP

Approaches to defining the genealogy of a CSBP:

- ▶ (Bertoin and Le Gall, 2000): Construct a three-parameter process $(S^{(s,t)}(a), 0 \leq s \leq t, a \geq 0)$, such that $S^{(s,t)}(a)$ is the number of individuals at time t descended from the first a individuals at time s . If $0 < y < X(t)$, then

$$\inf\{b : S^{(s,t)}(b) \geq y\}$$

is the ancestor of y at time s .

- ▶ (Donnelly and Kurtz, 1999): Represent the population by a countable system of particles. Follow the genealogy of the particles labelled $1, \dots, n$.

Describing the genealogy of a CSBP

Results for the genealogy of CSBPs:

- ▶ (Perkins, 1991): When $\Psi(\lambda) = c\lambda^2$, after a time-change, the genealogy is given by Kingman's coalescent.
- ▶ (Bertoin and Le Gall, 2000): When $\Psi(\lambda) = \lambda \log \lambda$, which is Neveu's CSBP, the genealogy is the Bolthausen-Sznitman coalescent.
- ▶ (Birkner, Blath, Capaldo, Etheridge, Möhle, Schweinsberg, and Wakolbinger, 2005): Consider a CSBP with $\sigma^2 = 0$ and $\nu(dx) = cx^{-1-\alpha} dx$, where $0 < \alpha < 2$. After a time-change, the genealogy is given by the Beta($2 - \alpha, \alpha$)-coalescent.

Comparison with fixed-size population model:

- ▶ Feller's branching diffusion is a limit of critical Bienaymé-Galton-Watson processes whose offspring distribution has finite variance.
- ▶ When $1 < \alpha < 2$, the α -stable CSBP is the limit of critical Bienaymé-Galton-Watson processes for which $P(L \geq k) \sim Ck^{-\alpha}$.

Obtaining the genealogy from the Lévy measure

Suppose $\nu(dx) = cx^{-1-\alpha} dx$. Let A be current population size.

After a jump of size x , a fraction $p = x/(A+x)$ of the population was born at the time of the jump. Tracing ancestral lines backwards in time, a p -merger occurs at this time.

Since $x/(A+x) \geq r$ if and only if $x \geq Ar/(1-r)$, we get p -mergers with $p \geq r$ at rate

$$A \int_{Ar/(1-r)}^{\infty} cx^{-1-\alpha} dx = \frac{cA^{1-\alpha}}{\alpha} \left(\frac{r}{1-r} \right)^{-\alpha}.$$

The rate of such mergers in the Beta($2-\alpha, \alpha$)-coalescent is

$$\frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_r^1 p^{-1-\alpha}(1-p)^{\alpha-1} dp = \frac{1}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)} \left(\frac{r}{1-r} \right)^{-\alpha}.$$

Populations with selection: Model 1

Introduced by Brunet, Derrida, Mueller, and Munier (2006, 2007):

- ▶ The population has fixed size N .
- ▶ Generations do not overlap.
- ▶ Each individual has $k \geq 2$ offspring.
- ▶ The fitness of each offspring is the parent's fitness plus an independent random variable with distribution ν .
- ▶ Of the kN offspring, the N with the highest fitness survive to form the next generation.

Populations with selection: Model 2

Model of a population in which individuals can repeatedly acquire beneficial mutations:

- ▶ The population has N individuals.
- ▶ Each individual independently acquires mutations at rate μ_N .
- ▶ Mutations are beneficial. An individual with j mutations at time t has fitness

$$\max\{1 + s_N(j - M(t)), 0\},$$

where $M(t)$ is the average number of mutations of the N individuals at time t . ($s_N =$ fitness benefit from a mutation)

- ▶ Each individual independently lives for an exponential(1) time.
- ▶ When an individual dies, its replacement is chosen at random from the population, with probability proportional to fitness.

Branching Brownian motion (BBM) approximations

Begin with some configuration of particles on \mathbb{R} at time zero.

Think of each particle as being an individual in a population, and position of the particle is the fitness.

Each particle independently moves according to one-dimensional Brownian motion with drift $-\rho$.

Approximation to Model 1 (BBM with absorption):

- ▶ Each particle splits into two at rate 1.
- ▶ Particles begin in $(0, \infty)$ and are killed when they reach 0.

Approximation to Model 2 (BBM with an inhomogeneous branching rate):

- ▶ Each particle dies at rate 1.
- ▶ A particle at x splits into two at rate $b(x) = 1 + \beta x$.

Modeling the evolution of populations undergoing selection using branching Brownian motion

Lecture II

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Review of Lecture I

For many classical population models (such as the Wright-Fisher model and the Moran model), the genealogy is given by Kingman's coalescent.

For populations with large family sizes, the genealogy may be described better by a coalescent with multiple mergers. The Beta($2 - \alpha, \alpha$)-coalescent arises when individuals have k or more offspring with probability proportional to $k^{-\alpha}$ with $1 \leq \alpha < 2$.

We introduced two population models involving selection. One can be approximated by BBM with absorption. The other can be approximated by BBM with an inhomogeneous branching rate.

Outline of the lecture

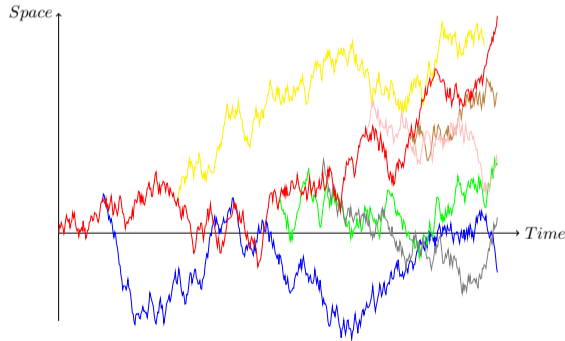
1. Ordinary BBM (without absorption).
 - ▶ Connection with FKPP equation
 - ▶ Many-to-one lemma
 - ▶ Additive and derivative martingales
 - ▶ Position of the right-most particle
2. BBM with absorption.
 - ▶ Moment calculations
 - ▶ The supercritical case
 - ▶ The subcritical case

Branching Brownian motion (BBM)

Begin with one particle at 0.

Each particle independently moves according to standard one-dimensional Brownian motion (without drift).

Each particle splits into two at rate 1.



Picture by Jiaqi Liu

Connection with the FKPP equation

Theorem (McKean, 1975): Let $R(t)$ be the position of the right-most particle at time t . Let $u(t, x) = P(R(t) \leq x)$. Then

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + u^2 - u, \quad u(0, x) = \mathbb{1}_{\{x \geq 0\}}.$$

This is the FKPP equation, introduced by Fisher (1937) and by Kolmogorov, Petrovskii, and Piskounov (1937).

Theorem (Kolmogorov, Petrovskii, and Piskounov, 1937). The FKPP equation has a traveling wave solution

$$u(t, x) = w(x - \sqrt{2} t),$$

where w satisfies

$$\frac{1}{2} w'' + \sqrt{2} w' + w^2 - w = 0.$$

If $m(t)$ is the median position of the right-most particle, $\lim_{t \rightarrow \infty} t^{-1} m(t) = \sqrt{2}$.

Explanation of connection between BBM and FKPP

Compute $u(t+h, x) = P(R(t+h) \leq x)$ for small h by conditioning at time h :

- ▶ With probability approximately h , the particle splits into two before time h . Then the probability that $R(t+h) \leq x$ is approximately $u(t, x)^2$.
- ▶ With probability approximately $1-h$, there is no branching by time h . If B_h is the position of the particle at time h , then the probability that $R(t+h) \leq x$ is approximately $E[u(t, x - B_h)]$.

Let $W_t = x - B_t$. If f is twice differentiable, then for small h ,

$$f(W_h) \approx f(x) + f'(x)(W_h - x) + \frac{1}{2}f''(x)(W_h - x)^2.$$

Taking expectations, $E[f(W_h)] \approx f(x) + \frac{1}{2}f''(x)h$. Therefore,

$$\begin{aligned}u(t+h, x) &\approx hu(t, x)^2 + (1-h)\left(u(t, x) + \frac{h}{2}\frac{\partial^2}{\partial x^2}u(t, x)\right) \\ &\approx u(t, x) + h\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}u(t, x) + u(t, x)^2 - u(t, x)\right).\end{aligned}$$

Many-to-one Lemma

Let $(B(t), t \geq 0)$ be Brownian motion started at x . Then $B(t) \sim N(x, t)$, and $B(t)$ has density

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/(2t)},$$

which satisfies the heat equation

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial y^2}.$$

For branching Brownian motion, let $N(t)$ be the number of particles at time t . We have $E[N(t)] = e^t$. The “density” for branching Brownian motion is given by

$$q_t(x, y) = e^t p_t(x, y).$$

If $A \subset \mathbb{R}$, then if there is one particle at x at time zero, the expected number of particles in A at time t is

$$\int_A q_t(x, y) dy.$$

Additive Martingales

Let $X_1(t) \geq X_2(t) \geq \dots \geq X_{N(t)}(t)$ be the positions of the particles at time t . Given $\lambda \in \mathbb{R}$, define

$$Y_\lambda(t) = \sum_{i=1}^{N(t)} e^{\lambda X_i(t) - (1 + \lambda^2/2)t}.$$

Proposition (McKean, 1975): $(Y_\lambda(t), t \geq 0)$ is a martingale.

Proof: Suppose we start with one particle at x . By the many-to-one Lemma,

$$\begin{aligned} E_x[Y_\lambda(t)] &= \int_{-\infty}^{\infty} q_t(x, y) e^{\lambda y - (1 + \lambda^2/2)t} dy \\ &= \int_{-\infty}^{\infty} p_t(x, y) e^{\lambda y - \lambda^2 t/2} dy = e^{\lambda x} = Y_\lambda(0). \end{aligned}$$

By the branching and Markov properties, $E_x[Y_\lambda(t) | \mathcal{F}_s] = Y_\lambda(s)$, so $(Y_\lambda(t), t \geq 0)$ is a martingale.

Derivative martingale (Lalley and Sellke, 1987)

We have $Y_\lambda(t) \rightarrow Y_\lambda$ as $t \rightarrow \infty$. If $|\lambda| < \sqrt{2}$, then $Y_\lambda > 0$ almost surely. If $|\lambda| \geq \sqrt{2}$, then $Y_\lambda = 0$ almost surely.

Let

$$Z(t) = \sum_{i=1}^{N(t)} (\sqrt{2}t - X_i(t)) e^{\sqrt{2}X_i(t) - 2t}.$$

Then $(Z(t), t \geq 0)$ is a martingale and $Z(t) \rightarrow Z$ as $t \rightarrow \infty$ with $P(Z > 0) = 1$.

Aïdékon and Shi (2014) proved that $\sqrt{t}Y_{\sqrt{2}}(t) \rightarrow CZ$ as $t \rightarrow \infty$.

Boutaud and Maillard (2019) gave an alternative proof.

Expected number of particles above a level

The expected number of particles above a at time t is

$$\int_a^\infty q_t(0, y) dy = e^t \int_a^\infty \frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)} dy \approx e^t \cdot \frac{\sqrt{t}}{a\sqrt{2\pi}} e^{-a^2/(2t)}.$$

If

$$a = \sqrt{2} t - \frac{1}{2\sqrt{2}} \log t,$$

then this expectation will converge to a constant as $t \rightarrow \infty$. By Markov's Inequality, the median position $m(t)$ of the right-most particle can not be above

$$\sqrt{2} t - \frac{1}{2\sqrt{2}} \log t + O(1).$$

This is an overestimate of $m(t)$. The number of particles near this level is usually zero, but is large with small probability.

Location of right-most particle

Bramson (1983) showed that the median position $m(t)$ of the right-most particle is:

$$m(t) = \sqrt{2} t - \frac{3}{2\sqrt{2}} \log t + O(1).$$

Theorem (Lalley and Sellke, 1987): Let $R(t)$ be the position of the right-most particle at time t . Then

$$\lim_{t \rightarrow \infty} P(R(t) \leq m(t) + x) = E[\exp(-CZe^{-\sqrt{2}x})],$$

where $C > 0$ is a constant and Z is the limit of the derivative martingale.

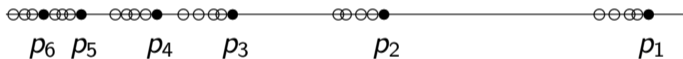
That is, conditional on Z , the asymptotic distribution of $R(t)$ is the Gumbel distribution shifted by $\frac{1}{\sqrt{2}} \log Z$.

The extremal point process

Theorem (Arguin, Bovier, and Kistler, 2013; Aidékon, Berestycki, Brunet, and Shi, 2013): Let

$$\mathcal{E}_t = \sum_{i=1}^{N(t)} \delta_{X_i(t) - m(t)}.$$

Conditional on Z , let \mathcal{P} be a Poisson point process with intensity $\sqrt{2}CZe^{-\sqrt{2}x} dx$. Let $(p_i)_{i=1}^{\infty}$ be the points of \mathcal{P} .



There is a probability measure ν on the set of point processes on $(-\infty, 0]$ such that if $(\Delta_i)_{i=1}^{\infty}$ is a sequence of independent point processes with distribution ν , then if we denote the points of Δ_i by $(q_{i,j})_{j=1}^{\infty}$ we have $\mathcal{E}_t \Rightarrow \mathcal{E}$ with

$$\mathcal{E} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{p_i + q_{i,j}}.$$

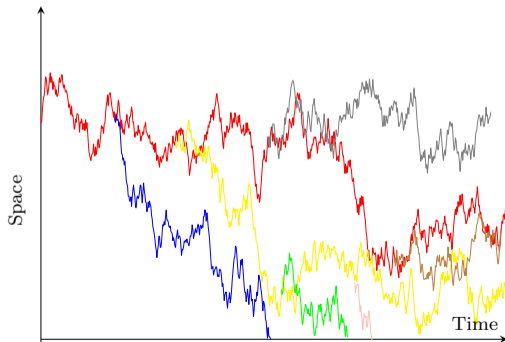
Branching Brownian motion with absorption

Begin with some configuration of particles in $(0, \infty)$.

Each particle independently moves according to standard one-dimensional Brownian motion with drift $-\rho$.

Each particle splits into two at rate 1.

Particles are killed if they reach the origin.



Picture by Jiaqi Liu

Condition for extinction

Theorem (Kesten, 1978): Branching Brownian motion with absorption dies out almost surely if $\rho \geq \sqrt{2}$. If $\rho < \sqrt{2}$, the process survives forever with positive probability.

We will consider five cases:

1. Supercritical case: $\rho < \sqrt{2}$.
2. Slightly supercritical case: $\rho = \sqrt{2 - \varepsilon}$.
3. Critical case: $\rho = \sqrt{2}$.
4. Slightly subcritical case: $\rho = \sqrt{2 + \varepsilon}$.
5. Subcritical case: $\rho > \sqrt{2}$.

Slightly supercritical case is most relevant for modeling populations of nearly fixed size.

In this lecture, we will focus on supercritical and subcritical cases.

First moment calculations

Consider a single Brownian particle started at x , with drift of $-\rho$ and absorption at 0. The “density” of the position of the particle at time t is

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} \left(e^{-(x-y)^2/2t} - e^{-(x+y)^2/2t} \right) \cdot e^{\rho x - \rho y - \rho^2 t/2}.$$

By the Many-to-one Lemma, for branching Brownian motion with absorption, the “density” is given by

$$q_t(x, y) = e^t p_t(x, y).$$

Let $(B(t), t \geq 0)$ be Brownian motion with drift $-\rho$ and absorption at zero. If $f : (0, \infty) \rightarrow \mathbb{R}$, then

$$E_x \left[\sum_{i=1}^{N(t)} f(X_i(t)) \right] = \int_0^\infty f(y) q_t(x, y) dy = e^t E[f(B(t))].$$

Take $f = \mathbb{1}_A$ to get expected number of particles in a set A .

First moment asymptotics

We have

$$q_t(x, y) = \frac{1}{\sqrt{2\pi t}} \left(e^{-(x-y)^2/2t} - e^{-(x+y)^2/2t} \right) \cdot e^{\rho x - \rho y + (1-\rho^2/2)t}.$$

Let $t \rightarrow \infty$ with x and y fixed:

$$q_t(x, y) \sim \frac{\sqrt{2}}{\sqrt{\pi} t^{3/2}} x e^{\rho x} y e^{-\rho y} e^{(1-\rho^2/2)t}$$

and

$$E_x[N(t)] \sim \frac{\sqrt{2}}{\sqrt{\pi} t^{3/2} \rho^2} x e^{\rho x} e^{(1-\rho^2/2)t}.$$

Notes:

1. $q_t(x, y)$ is proportional to $y e^{-\rho y}$.
2. $q_t(x, y)$ and $E_x[N(t)]$ are proportional to $x e^{\rho x}$.
3. $E_x[N(t)] \rightarrow 0$ if and only if $\rho \geq \sqrt{2}$.

Second moment calculations

Theorem (Ikeda, Nagasawa, Watanabe, 1969): If $f : (0, \infty) \rightarrow \mathbb{R}$,

$$E_x \left[\left(\sum_{i=1}^{N(t)} f(X_i(t)) \right)^2 \right] = \int_0^\infty f(y)^2 q_t(x, y) dy \\ + 2 \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty f(y_1) f(y_2) q_s(x, z) q_{t-s}(z, y_1) q_{t-s}(z, y_2) dy_1 dy_2 dz ds.$$

Many-to-two lemma: Consider $(B_1(t), t \geq 0)$ and $(B_2(t), t \geq 0)$ evolving as Brownian motion with drift $-\rho$ and absorption at zero. The particles stay together until time $T \sim \text{Exponential}(2)$ and move independently after time T . Then

$$E_x \left[\left(\sum_{i=1}^{N(t)} f(X_i(t)) \right)^2 \right] = E_x \left[f(B_1(t)) f(B_2(t)) e^{2t + (T \wedge t)} \right].$$

Many-to-few lemma: Harris and Roberts (2017) generalize to k th moments.

A martingale

For all $t \geq 0$, let

$$V(t) = \sum_{i=1}^{N(t)} X_i(t) e^{\rho X_i(t) - (1 - \rho^2/2)t}.$$

By the many-to-one lemma, the Markov property, and the branching property, $(V(t), t \geq 0)$ is a martingale.

Theorem (Harris and Harris, 2007): We have $V(t) \rightarrow V$ almost surely as $t \rightarrow \infty$. If $\rho < \sqrt{2}$, then $(V(t), t \geq 0)$ is uniformly integrable, and $V > 0$ almost surely on the event of nonextinction.

The supercritical case

Consider the supercritical case with $\rho < \sqrt{2}$. Start with one particle at x .

Theorem: We have

$$\lim_{t \rightarrow \infty} \frac{N(t)}{E_x[N(t)]} = \frac{V}{xe^{\rho x}}.$$

Also, let ν be the probability measure with

$$\nu(dy) = \rho^2 ye^{-\rho y} \mathbb{1}_{(0, \infty)}(y) dy.$$

Almost surely on the event of nonextinction,

$$\lim_{t \rightarrow \infty} \frac{1}{N(t)} \sum_{i=1}^{N(t)} \delta_{X_i(t)} = \nu.$$

Kesten (1978) stated a very similar result but wrote, “So far we have only an ugly and complicated proof”.

Loudor and Saglietti (2020) provided a complete proof.

Idea of the proof ($\rho < 1$)

Let $N_t(A)$ be the number of particles in A at time t .

The main idea

- ▶ Calculate $E[N_t(A)]$ and $\text{Var}(N_t(A))$ using first and second moment calculations.
- ▶ Show that $\text{Var}(N_t(A)) \ll (E[N_t(A)])^2$ for large t .

The randomness at the beginning

- ▶ Because of randomness at the beginning of the process, we do not have $\text{Var}(N_t(A)) \ll (E[N_t(A)])^2$.
- ▶ Condition on \mathcal{F}_s and calculate $E[N_t(A)|\mathcal{F}_s]$ and $\text{Var}(N_t(A)|\mathcal{F}_s)$.
- ▶ $E[N_t(A)|\mathcal{F}_s]$ is proportional to $V(s)$, which is approximately V when s is large.
- ▶ When s is large, we have $\text{Var}(N_t(A)|\mathcal{F}_s) \ll (E[N_t(A)|\mathcal{F}_s])^2$.

The need for truncation when $1 \leq \rho < \sqrt{2}$

(Jonckheere and Saglietti, 2020): Convergence in L^2 only holds when $\rho < 1$.

When $1 \leq \rho < \sqrt{2}$, the second moment is dominated by rare events in which one particle gets unusually far to the right, and then has many surviving descendants.

Consider the process in which particles are killed if they reach the curve $u \mapsto M(1 + u^{3/4})$.

Show that this truncation has minimal effect on $E_x[N_t(A)]$.

Show that for the process with truncation,

$$\text{Var}(N_t(A)|\mathcal{F}_s) \ll (E[N_t(A)|\mathcal{F}_s])^2.$$

The subcritical case

Consider the subcritical case with $\rho > \sqrt{2}$.

Suppose we start with one particle at x .

Let $\zeta = \inf\{t : N(t) > 0\}$ be the extinction time.

Theorem (Harris and Harris, 2007): There is a constant K such that as $t \rightarrow \infty$,

$$P_x(\zeta > t) \sim \frac{K}{\sqrt{2\pi} t^{3/2}} x e^{\rho x + (1 - \rho^2/2)t} \sim \frac{\rho^2 K}{2} E_x[N(t)],$$

where K depends on ρ but not on x . In particular,

$$\lim_{t \rightarrow \infty} E_x[N(t) | \zeta > t] = \frac{2}{\rho^2 K}.$$

Remark: Harris and Harris (2007) observe that, as a consequence of arguments in Chauvin and Rouault (1988), there is a probability distribution $(\pi_j)_{j=1}^{\infty}$ with mean $2/(\rho^2 K)$ such that

$$\lim_{t \rightarrow \infty} P_x(N(t) = j | \zeta > t) = \pi_j.$$

The slightly subcritical case

Consider the subcritical case with $\rho = \sqrt{2 + 2\varepsilon}$.

Suppose we start with one particle at x .

Theorem (Liu, 2021): There are positive constants C_1 and C_2 such that for sufficiently small $\varepsilon > 0$,

$$e^{C_1/\sqrt{\varepsilon}} \leq \lim_{t \rightarrow \infty} E_x[N(t) \mid \zeta > t] \leq e^{C_2/\sqrt{\varepsilon}}.$$

This conditional expectation is likely dominated by rare events. We conjecture that, conditional on $\zeta > t$, typically $\log N(t)$ is $O(\varepsilon^{-1/3})$ rather than $O(\varepsilon^{-1/2})$.

Spine decomposition (Chauvin and Rouault, 1988; Harris and Harris, 2007)

Recall we have a martingale defined by

$$V(t) = \sum_{i=1}^{N(t)} X_i(t) e^{\rho X_i(t) - (1-\rho^2/2)t}.$$

Define a probability measure Q_x by

$$\left. \frac{dQ_x}{dP_x} \right|_{\mathcal{F}_t} = \frac{V(t)}{V(0)}.$$

Under Q_x , the BBM evolves as follows:

- ▶ There is a distinguished particle, called the spine, which evolves as a Bessel(3) process started from x .
- ▶ The spine gives birth to new particles at rate 2 (twice the rate).
- ▶ Particles that branch off the spine initiate independent copies of BBM with drift.

For $A \in \mathcal{F}_s$, we have $\lim_{t \rightarrow \infty} P_x(A | \zeta > t) = Q_x(A)$.

Idea of the proof in the slightly subcritical case

From the result of Harris and Harris (2007),

$$K = \lim_{t \rightarrow \infty} \frac{\sqrt{2\pi} t^{3/2}}{x e^{\rho x}} e^{-(1-\rho^2/2)t} P_x(\zeta > t)$$

Now using the change of measure (and using Q_x to denote both probabilities and expectations)

$$\begin{aligned} K &= \lim_{t \rightarrow \infty} \frac{\sqrt{2\pi} t^{3/2}}{x e^{\rho x}} e^{-(1-\rho^2/2)t} Q_x \left(\frac{V(0)}{V(t)} \mathbb{1}_{\{\zeta > t\}} \right) \\ &= \lim_{t \rightarrow \infty} \sqrt{2\pi} t^{3/2} Q_x \left(\frac{1}{\sum_{i=1}^{N(t)} X_i(t) e^{\rho X_i(t)}} \right). \end{aligned}$$

One can then use the spine decomposition to estimate K .

Modeling the evolution of populations undergoing selection using branching Brownian motion

Lecture III

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Populations with selection: Model 1

Introduced by Brunet, Derrida, Mueller, and Munier (2006, 2007):

- ▶ The population has fixed size N .
- ▶ Generations do not overlap.
- ▶ Each individual has $k \geq 2$ offspring.
- ▶ The fitness of each offspring is the parent's fitness plus an independent random variable with distribution ν .
- ▶ Of the kN offspring, the N with the highest fitness survive to form the next generation.

Three conjectures (Brunet, Derrida, Mueller, and Munier (2006, 2007))

1. (Brunet and Derrida, 1997) Let R_m be the maximum of the fitnesses of the individuals in generation m . Then $R_m/m \rightarrow v_N$ a.s. Let $v_\infty = \lim_{N \rightarrow \infty} v_N$. Then

$$v_\infty - v_N \sim \frac{C}{(\log N)^2}.$$

Note: Bérard and Gouéré (2010) proved this conjecture in the case $k = 2$ under suitable regularity conditions on ν .

2. If two individuals are chosen from some generation, then the number of generations back to their most recent common ancestor is $O((\log N)^3)$.
3. If n individuals are sampled from some generation, then the coalescence of the ancestral lineages is governed by the Bolthausen-Sznitman coalescent.

We will establish rigorous versions of these conjectures for branching Brownian motion with absorption.

A simple model with a Bolthausen-Sznitman genealogy

Population model considered by Brunet, Derrida, Mueller, and Munier (2007):

- ▶ The population has fixed size N , and generations do not overlap.
- ▶ An individual located at x produces offspring at $(x + Y_i)_{i=1}^{\infty}$ in the next generation, where $(Y_i)_{i=1}^{\infty}$ are points of a Poisson process on \mathbb{R} with intensity $e^{-y} dy$.
- ▶ The N right-most particles are selected to form the next generation.

Offspring of particle at x are points of a Poisson process with intensity $e^x e^{-y} dy$.

If the N individuals are located at x_1, \dots, x_N , then the probability that an individual at y in the next generation is descended from the individual at x_i is

$$\frac{e^{x_i}}{\sum_{k=1}^N e^{x_k}}.$$

The probability that there is an individual with $e^{x_i} > z$, or $x_i > \log z$, is proportional to $e^{-\log z} = z^{-1}$, leading to a Bolthausen-Sznitman genealogy.

Branching Brownian motion with absorption

Begin with some configuration of particles in $(0, \infty)$.

Each particle independently moves according to standard one-dimensional Brownian motion with drift $-\rho$.

Each particle splits into two at rate 1.

Particles are killed if they reach the origin.

Theorem (Kesten, 1978): BBM with absorption dies out almost surely if $\rho \geq \sqrt{2}$. If $\rho < \sqrt{2}$, the process survives forever with positive probability.

In the previous lecture, we considered cases of supercritical drift ($\rho < \sqrt{2}$) and subcritical drift ($\rho > \sqrt{2}$).

We want to approximate a population of size $O(N)$ using BBM with absorption. Consider the slightly supercritical case, in which ρ is slightly smaller than $\sqrt{2}$.

Based on joint work with Julien Berestycki and Nathanaël Berestycki.

The choice of ρ and Conjecture 1

To keep the population size $O(N)$ for a long time, we choose

$$\rho = \sqrt{2 - \frac{2\pi^2}{(\log N + 3 \log \log N)^2}} = \sqrt{2} - O\left(\frac{1}{(\log N)^2}\right).$$

Our model is equivalent to one in which there is no drift and, at time t , particles are killed if they reach ρt .

That is, ρ can be interpreted as the speed at which the fitness of the population increases.

That the correction term is $O((\log N)^{-2})$ is related to Conjecture 1.

Some notation

Let $M_N(t)$ be the number of particles at time t .

Let $X_{1,N}(t) \geq X_{2,N}(t) \geq \dots \geq X_{M_N(t),N}(t)$ be the positions of the particles at time t .

Let

$$L = \frac{1}{\sqrt{2}} \left(\log N + 3 \log \log N \right)$$
$$Y_N(t) = \sum_{i=1}^{M_N(t)} e^{\rho X_{i,N}(t)}$$
$$Z_N(t) = \sum_{i=1}^{M_N(t)} e^{\rho X_{i,N}(t)} \sin \left(\frac{\pi X_{i,N}(t)}{L} \right) \mathbb{1}_{\{X_{i,N}(t) \leq L\}}$$

$Z_N(t)$ will be a useful measure of the “size” of the process at time t , disregarding particles to the right of L .

Convergence to Neveu's CSBP

Theorem (Berestycki, Berestycki, Schweinsberg, 2013): Suppose

$$\frac{Z_N(0)}{N(\log N)^2} \Rightarrow \nu, \quad \frac{Y_N(0)}{N(\log N)^3} \Rightarrow 0.$$

For some $a \in \mathbb{R}$, the finite-dimensional distributions of

$$\left(\frac{1}{2\pi N} M_N((\log N)^3 t), t > 0 \right)$$

converge as $N \rightarrow \infty$ to those of the CSBP with initial distribution ν and branching mechanism $\Psi(u) = au + 2\pi^2 u \log u$.

Genealogy of the population

Theorem (Berestycki, Berestycki, Schweinsberg, 2013): Suppose

$$\frac{Z_N(0)}{N(\log N)^2} \Rightarrow \nu \quad \text{with } \nu(\{0\}) = 0, \quad \frac{Y_N(0)}{N(\log N)^3} \Rightarrow 0.$$

Choose $t > 0$, and pick n particles at random at time $t(\log N)^3$. Let $\Psi_N(s)$ be the partition of $\{1, \dots, n\}$ such that i and j are in the same block if and only if the i th and j th sampled particles have the same ancestor at time $(t - s/2\pi)(\log N)^3$. The finite-dimensional distributions of $(\Psi_N(s), 0 \leq s \leq 2\pi t)$ converge as $N \rightarrow \infty$ to those of the Bolthausen-Sznitman coalescent.

Explanation for the multiple mergers

Occasionally, a particle gets very far to the right. This particle has a large number of surviving descendants, as the descendants avoid the barrier at zero.

This leads to sudden jumps in the number of particles, and multiple mergers of ancestral lines.

Proof strategy:

- ▶ Find the level L such that a particle must reach L to give rise to a large jump in the number of particles.
- ▶ Show that the behavior of branching Brownian motion with particles killed at 0 and L is approximately deterministic. (This is a “Law of Large Numbers” result which is proved by calculating first and second moments.)
- ▶ Separately determine the (random) contribution of the particles that reach L .

Branching Brownian motion in a strip

Consider Brownian motion started at x , killed at 0 and L .

The “density” $p_t(x, y)$ of the position of the Brownian particle at time t satisfies

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial y^2}, \quad p_t(x, 0) = 0, \quad p_t(x, L) = 0.$$

Look for solutions of the form $e^{\lambda t} h(y)$, which must satisfy

$$\lambda h(y) = \frac{1}{2} h''(y), \quad h(0) = 0, \quad h(L) = 0.$$

Solutions are

$$h(y) = \sin\left(\frac{n\pi y}{L}\right), \quad \lambda = \frac{\pi^2 n^2}{2L^2}.$$

Therefore, the density has the form

$$p_t(x, y) = \sum_{n=1}^{\infty} a_n e^{-\pi^2 n^2 t / 2L^2} \sin\left(\frac{n\pi y}{L}\right).$$

Derivation of density formula

We have

$$p_t(x, y) = \sum_{n=1}^{\infty} a_n e^{-\pi^2 n^2 t / 2L^2} \sin\left(\frac{n\pi y}{L}\right).$$

The density must converge to a unit mass at x as $t \rightarrow 0$. If the initial particle had density f on $(0, L)$, then

$$f(y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{L}\right).$$

By orthogonality,

$$\int_0^L f(y) \sin\left(\frac{n\pi y}{L}\right) dy = a_n \int_0^L \sin^2\left(\frac{n\pi y}{L}\right) dy = \frac{La_n}{2}.$$

Therefore, taking $f(y)$ to be a unit mass at x , we get

$$a_n = \frac{2}{L} \sin\left(\frac{n\pi x}{L}\right).$$

Adding a drift and branching

We have

$$p_t(x, y) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t / 2L^2} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) dy.$$

Add branching at rate 1 and a drift of $-\rho$, the “density” for BBM in a strip is:

$$q_t(x, y) = p_t(x, y) \cdot e^{\rho(x-y) - \rho^2 t / 2} \cdot e^t,$$

meaning that if $A \subset (0, L)$, the expected number of particles in A at time t is

$$\int_A q_t(x, y) dy.$$

Alternatively, one could start from

$$\frac{\partial q}{\partial t} = \frac{1}{2} \frac{\partial^2 q}{\partial y^2} + \rho \frac{\partial q}{\partial y} + q.$$

For $t \gg L^2$,

$$q_t(x, y) \approx \frac{2}{L} e^{(1 - \rho^2 / 2 - \pi^2 / 2L^2)t} e^{\rho x} \sin\left(\frac{\pi x}{L}\right) e^{-\rho y} \sin\left(\frac{\pi y}{L}\right).$$

Observations related to the density formula

$$q_t(x, y) \approx \frac{2}{L} e^{(1-\rho^2/2-\pi^2/2L^2)t} e^{\rho x} \sin\left(\frac{\pi x}{L}\right) e^{-\rho y} \sin\left(\frac{\pi y}{L}\right).$$

- ▶ When

$$1 - \frac{\rho^2}{2} - \frac{\pi^2}{2L^2} = 0,$$

the density formula does not depend on t . We choose ρ to satisfy this equation, to keep the number of particles stable.

- ▶ The density formula is proportional to $e^{\rho x} \sin(\pi x/L)$. Summing over multiple particles at time 0, this becomes $Z_N(0)$. Thus, $Z_N(0)$ determines how many particles will be in a given set at future times.
- ▶ When ρ is chosen as above and particles are killed at 0 and L , the process $(Z_N(t), t \geq 0)$ is a martingale.
- ▶ The density formula is proportional to $e^{-\rho y} \sin(\pi y/L)$. Thus, for $t \gg (\log N)^2$, particles settle into a stable limiting configuration.

The choice of L

Suppose we place N particles on $(0, L)$ at random with density

$$g(y) = CLe^{-\rho y} \sin\left(\frac{\pi y}{L}\right).$$

We have $Z_N(0) = O(NL^2)$.

The contribution to Z_N from a particle at $L - 1$ is

$$e^{\rho(L-1)} \sin\left(\frac{\pi(L-1)}{L}\right) = O(L^{-1}e^{\rho L}),$$

which is comparable to NL^2 when

$$L = \frac{1}{\sqrt{2}} \left(\log N + 3 \log \log N \right).$$

The $(\log N)^3$ time scale

Suppose we place N particles on $(0, L)$ at random with density

$$g(y) = CLe^{-\rho y} \sin\left(\frac{\pi y}{L}\right).$$

A particle between $L - 1$ and L has positive probability of hitting L by time 1.

Expected number of particles in $(L - 1, L)$ is approximately

$$N \int_{L-1}^L CLe^{-\rho y} \sin\left(\frac{\pi y}{L}\right) dy = O((\log N)^{-3}).$$

We have to wait $O((\log N)^3)$ time for a particle to reach L .

Beyond truncated moment calculations

When particles are killed at 0 and at L :

- ▶ Second moment is too large to conclude that the number of particles in the system stays close to its expectation.
- ▶ The probability that a particle is killed at L does not tend to zero.

Idea: kill particles instead at $L + A$:

- ▶ Let $A \rightarrow -\infty$, and then the number of particles stays close to its expectation.
- ▶ Let $A \rightarrow \infty$, and then the probability that a particle hits the right boundary tends to zero.

Because we can't do both, proceed as follows:

- ▶ Stop particles when they reach $L - A$, for large A .
- ▶ After a particle hits $L - A$, follow the descendants of this particle until they reach $L - A - y$ for large y . Then re-incorporate them into the process.

The particles after they reach $L - A$ (Neveu, 1987)

Consider BBM with drift $-\sqrt{2}$ started with one particle at $L - A$.

Let $M(y)$ be the number of particles that reach $L - A - y$, if particles are killed upon reaching $L - A - y$.

Conditional on $M(x)$, the distribution of $M(x + y)$ is that of $M(x)$ independent random variables with the same distribution as $M(y)$, so $(M(y), y \geq 0)$ is a continuous-time branching process.

Offspring distribution has finite mean but is not in $L \log L$ class.

A limit theorem for the branching process

Theorem (Neveu, 1987): There exists a random variable W such that almost surely

$$\lim_{y \rightarrow \infty} ye^{-\sqrt{2}y} M(y) = W.$$

Furthermore, for all $u \in \mathbb{R}$,

$$E[e^{-e^{\sqrt{2}u}W}] = \psi(u),$$

where ψ satisfies $\frac{1}{2}\psi'' - \sqrt{2}\psi' = \psi(1 - \psi)$.

Proposition (Buraczewski, 2009; Maillard, 2012; Berestycki, Berestycki, Schweinsberg, 2013): As $x \rightarrow \infty$, we have

$$P(W > x) \sim \frac{1}{x\sqrt{2}}.$$

Note: The random variable W is the limit of the derivative martingale.

Connection with Neveu's CSBP and Bolthausen-Sznitman

Waiting time for a particle to hit L is $O((\log N)^3)$.

Rate at which particles reach L is proportional to $Z_N(t)$.

If a particle hits L , its contribution is proportional to the number of descendants that hit $L - y$ for large y , and therefore is approximately proportional to W .

The probability that $Z_N(t)/(N(\log N)^2)$ jumps by at least x is roughly Cx^{-1} .

For any CSBP, the rate of jumps of size at least x is proportional to the value of the process.

For Neveu's CSBP with $\nu(dy) = y^{-2} dy$, the rate of jumps of size at least x is proportional to $\int_x^\infty y^{-2} dy = x^{-1}$.

Configuration of particles

Idea: “density” of particles at $y \in (0, L)$ is proportional to $e^{-\rho y} \sin(\pi y/L)$.

Theorem (Berestycki, Berestycki, Schweinsberg, 2015): Let $t = s(\log N)^3$, where $0 < s < \infty$. Let

$$\chi(t) = \frac{1}{M_N(t)} \sum_{i=1}^{M_N(t)} \delta_{X_i(t)}.$$

Then $\chi(t) \Rightarrow \mu$ as $N \rightarrow \infty$, where μ is the probability measure on $(0, \infty)$ whose density is $g(y) = 2ye^{-\sqrt{2}y}$.

Theorem (Berestycki, Berestycki, Schweinsberg, 2015): Let $t = s(\log N)^3$, where $0 < s < \infty$. Let

$$\eta(t) = \frac{1}{Y(t)} \sum_{i=1}^{M_N(t)} e^{\rho X_i(t)} \delta_{X_i(t)/L}, \quad Y(t) = \sum_{i=1}^{M_N(t)} e^{\rho X_i(t)}.$$

Then $\eta(t) \Rightarrow \nu$ as $N \rightarrow \infty$ where ν is the probability measure on $(0, 1)$ whose density is $h(y) = \frac{\pi}{2} \sin(\pi y)$.

BBM with N particles (Maillard, 2016)

Begin with N particles chosen from density $g(y) = Ce^{-\sqrt{2}y} \sin(\pi y/L) \mathbb{1}_{(0,L)}(y)$.

Each particle independently moves according to one-dimensional Brownian motion. Each particle gives birth at rate 1, and the left-most particle is killed.

Let

$$\rho = \sqrt{2 - \frac{2\pi^2}{(\log N + 3 \log \log N)^2}}.$$

For $0 < \alpha < 1$, let $Q_\alpha(t)$ be the α -quantile of the locations of the particles at time t . Let $M_\alpha(t) = Q_\alpha(t) - \rho t$.

Theorem (Maillard, 2016) The finite-dimensional distributions of $(M_\alpha((\log N)^3 t), t \geq 0)$ converge as $N \rightarrow \infty$ to those of a Lévy process $(Y(t), t \geq 0)$, whose Lévy measure is the image of the measure $x^{-2} \mathbb{1}_{(0,\infty)}(x) dx$ by the map $x \mapsto \log(1+x)$.

This result gives the $(\log N)^3$ time scale but not the genealogy.

More results for BBM with N particles

DeMasi, Ferrari, Presutti, Soprano-Loto (2019) considered the N -BBM starting with N particles with density f .

They showed that the empirical distribution of particle positions at time t converges as $N \rightarrow \infty$ to a probability measure with density $x \mapsto u(t, x)$, which solves the following free boundary problem, provided a solution exists:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + u && \text{in } (L_t, \infty) \\ u(0, x) &= f(x) \\ u(t, L_t) &= 0, \quad \int_{L_t}^{\infty} u(t, x) dx = 1, && \text{for all } t \geq 0 \end{aligned}$$

Berestycki, Brunet, and Penington (2019) showed existence of a solution to the free boundary problem.

Note: $u(t, x)$ gives the empirical density of particles for fixed t , as $N \rightarrow \infty$.

A Poisson process model (Cortines and Mallein, 2018)

Generalization of the model of Brunet, Derrida, Mueller, Munier (2007):

- ▶ The population has fixed size N , and generations do not overlap.
- ▶ Let $0 < a \leq 1$. An individual located at x produces offspring at $(ax + Y_i)_{i=1}^{\infty}$, where $(Y_i)_{i=1}^{\infty}$ are points of a Poisson process on \mathbb{R} with intensity $e^{-y} dy$.
- ▶ The N right-most particles are selected to form the next generation.

Offspring of particle at x are points of a Poisson process with intensity $e^{ax} e^{-y} dy$.

If the N individuals are located at x_1, \dots, x_N , then the probability that an individual at y in the next generation is descended from the individual at x_i is

$$\frac{e^{ax_i}}{\sum_{k=1}^N e^{ax_k}}.$$

Let $\alpha = 1/a$. The probability that there is an individual with $e^{ax_i} > z$, or $x_i > \alpha \log z$, is proportional to $e^{-\alpha \log z} = z^{-\alpha}$.

For $1 \leq \alpha < 2$, the genealogy should be the Beta($2 - \alpha, \alpha$) coalescent.

The beta coalescent genealogy

Theorem (Cortines and Mallein, 2018): Sample n individuals in generation T . Let $\Psi_N(k)$ be the partition of $\{1, \dots, n\}$ such that i and j are in the same block if and only if the i th and j th sampled individuals have the same ancestor in generation $T - k$. Then:

- ▶ If $0 < a < 1/2$, then $(\Psi_N(\lfloor Nt \rfloor), t \geq 0)$ converges to Kingman's coalescent.
- ▶ If $a = 1/2$, then $(\Psi_N(\lfloor Nt/(\log N) \rfloor), t \geq 0)$ converges to Kingman's coalescent.
- ▶ If $1/2 < a < 1$, then $(\Psi_N(\lfloor N^{\alpha-1}t \rfloor), t \geq 0)$ converges to the Beta($2 - \alpha, \alpha$) coalescent. (Recall $\alpha = 1/a$)
- ▶ (Brunet, Derrida, Mueller, and Munier, 2007): If $a = 1$, then $(\Psi_N(\lfloor (\log N)t \rfloor), t \geq 0)$ converges to the Bolthausen-Sznitman coalescent.

Cortines and Mallein (2018) conjectured that the beta coalescent could give the genealogy of the branching Ornstein-Uhlenbeck process, with selection of the N right-most particles.

A BBM model for semi-pushed fronts

Tourniaire (2021) consider the following branching Brownian motion model:

- ▶ At time zero, there are N particles located at 1.
- ▶ Particles independently move as Brownian motion with drift $-\rho$.
- ▶ Particles are killed if they reach the origin.
- ▶ Particles in $(0, 1]$ branch at rate $\beta/2$, where $\beta > 1$. Particles in $(1, \infty)$ branch at rate $1/2$.

The drift ρ depends on β and is chosen to make the process “critical”.

Allee effect: Population growth can be slowed by low population density.

Birzu, Hallatschek, Korolev (2018, 2021) argued that the Allee effect can produce beta coalescent genealogies, for example, when a population is expanding into a new geographic area.

Convergence to the α -stable CSBP

Theorem (Tourniaire, 2021): Let $M_N(t)$ be the number of particles at time t . There are constants $1 < \beta_1 < \beta_2$ such that if $\beta_1 < \beta < \beta_2$, then the finite-dimensional distributions of

$$\left(\frac{C}{N} M_N(N^{\alpha-1}t), t \geq 0 \right)$$

converge as $N \rightarrow \infty$ to the finite-dimensional distributions of an α -stable CSBP, where $C > 0$ and $\alpha \in (1, 2)$ depend on β .

We expect the genealogy of the population to be given by the Beta($2 - \alpha, \alpha$) coalescent.

Proof proceeds by stopping particles at L , following their descendants.

The particles after they reach L

Because the branching rate is lower near L , the drift (after scaling) is subcritical.

The next result is the counterpart to that of Neveu (1987) in the critical case.

Theorem (Chauvin, 1991; Liu, 2000; Tourniaire, 2021) Consider branching Brownian motion with drift $-\rho$, where $\rho > \sqrt{2}$, started with one particle at L . Let $M(y)$ be the number of particles that reach $L - y$, if particles are killed upon reaching $L - y$. There exists a random variable W such that almost surely

$$\lim_{y \rightarrow \infty} e^{-(\rho - \sqrt{\rho^2 - 2})y} M(y) = W,$$

where as $x \rightarrow \infty$,

$$P(W > x) \sim \frac{C}{x^\alpha}, \quad \alpha = \frac{\rho + \sqrt{\rho^2 - 2}}{\rho - \sqrt{\rho^2 - 2}}.$$

Modeling the evolution of populations undergoing selection using branching Brownian motion

Lecture IV

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Branching Brownian motion with absorption

Begin with some configuration of particles in $(0, \infty)$.

Each particle independently moves according to standard one-dimensional Brownian motion with drift $-\rho$.

Each particle splits into two at rate 1.

Particles are killed if they reach the origin.

We think of each particle as being an individual in a population, and the position of each particle is the fitness. Absorption at zero models the deaths of individuals whose fitness gets too low.

Theorem (Kesten, 1978): BBM with absorption dies out almost surely if $\rho \geq \sqrt{2}$. If $\rho < \sqrt{2}$, the process survives forever with positive probability.

Outline of the lecture

In Lecture II, we considered supercritical ($\rho < \sqrt{2}$) and subcritical ($\rho > \sqrt{2}$) drift.

In Lecture III, we used the case of slightly supercritical drift ($\rho = \sqrt{2 - \varepsilon}$) to model a population of approximately fixed size. We showed that the population size evolves over time like Neveu's CSBP, and the genealogy of the population can be described by a Bolthausen-Sznitman coalescent.

In this lecture, we will:

- ▶ Consider the critical case $\rho = \sqrt{2}$, and use the ideas from the slightly supercritical case, along with other tools, to improve upon earlier results.
Two questions:
 - ▶ What is the probability that the process survives until a large time t ?
 - ▶ Conditional on survival until a large time t , what does the configuration of particles look like at time t ? (Yaglom-type limit theorems)
- ▶ Consider the second population model introduced in the first lecture.

Long-run survival probability

Consider BBM with absorption, critical drift $\rho = \sqrt{2}$.

Let $N(t)$ be the number of particles at time t .

Let $\zeta = \inf\{t : N(t) = 0\}$ be the extinction time.

Theorem (Kesten, 1978): There exists $K > 0$ such that for each $x > 0$, we have for sufficiently large t :

$$xe^{\sqrt{2}x - (3\pi^2 t)^{1/3} - K(\log t)^2} \leq P_x(\zeta > t) \leq (1 + x)e^{\sqrt{2}x - (3\pi^2 t)^{1/3} + K(\log t)^2}.$$

Theorem (Maillard, Schweinsberg, 2020): There is a positive constant C such that for all $x > 0$, we have as $t \rightarrow \infty$,

$$P_x(\zeta > t) \sim Cxe^{\sqrt{2}x - (3\pi^2 t)^{1/3}}.$$

Derrida and Simon (2007) obtained this result nonrigorously.

The weaker bound $C_1xe^{\sqrt{2}x - (3\pi^2 t)^{1/3}} \leq P_x(\zeta > t) \leq C_2xe^{\sqrt{2}x - (3\pi^2 t)^{1/3}}$ was obtained by Berestycki, Berestycki, Schweinsberg (2014).

The process conditioned on survival

Let $N(t)$ be the number of particles at time t .

Let $R(t)$ be the position of the right-most particle at time t .

Theorem (Kesten, 1978): There are positive constants K_1 and K_2 such that for all $x > 0$,

$$\lim_{t \rightarrow \infty} P_x(N(t) > e^{K_1 t^{2/9} (\log t)^{2/3}} \mid \zeta > t) = 0$$

$$\lim_{t \rightarrow \infty} P_x(R(t) > K_2 t^{2/9} (\log t)^{2/3} \mid \zeta > t) = 0.$$

Theorem (Maillard, Schweinsberg, 2020): If the process starts with one particle at $x > 0$, conditional on survival until time t ,

$$t^{-2/9} \log N(t) \Rightarrow a_1 V^{1/3}$$

$$t^{-2/9} R(t) \Rightarrow a_2 V^{1/3},$$

for some constants $a_1 > 0$ and $a_2 > 0$, where V has an exponential distribution.

A curved right boundary

Recall that the density for the process with particles killed at 0 and L is

$$q_t(x, y) \approx \frac{2}{L} e^{-(\pi^2/2L^2)t} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L}\right).$$

Because the number of particles decreases over time, we need to move the right boundary closer to the origin over time.

Fix $t > 0$. Let $c = (3\pi^2)^{1/3}/\sqrt{2}$. Let $L_t(s) = c(t - s)^{1/3}$ for $s \in [0, t]$.

We will kill particles when they reach the curve $s \mapsto L_t(s)$.

This right boundary was previously used by Kesten (1978).

Roughly, a particle that gets within a constant of $L_t(s)$ at time s has a good chance to have a descendant alive at time t .

Deriving the right boundary

The number of particles approximately follows

$$\frac{d}{ds} N(s) \approx -\frac{\pi^2}{2L_t(s)^2} N(s).$$

We want to choose $s \mapsto L_t(s)$ so that

$$L_t(s) \approx \frac{1}{\sqrt{2}} \log N(s).$$

This leads to

$$\frac{d}{ds} L_t(s) = \frac{1}{\sqrt{2}} \frac{\frac{d}{ds} N(s)}{N(s)} = -\frac{\pi^2}{2\sqrt{2} L_t(s)^2}, \quad L_t(t) = 0,$$

which has the solution

$$L_t(s) = c(t - s)^{1/3}, \quad c = \frac{(3\pi^2)^{1/3}}{\sqrt{2}}.$$

Estimating the density and applying truncation

Use methods of Novikov (1981) and Roberts (2012) to approximate the density.

Density formula resembles that for BBM in a strip:

$$q_s(x, y) \approx \frac{2}{\sqrt{L_t(0)L_t(s)}} e^{-\sqrt{2}(L_t(0)-L_t(s))} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L_t(0)}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi x}{L_t(s)}\right).$$

We proceed as follows:

- ▶ Stop particles when they reach $L_t(s) - A$, for large A .
- ▶ After a particle hits $L_t(s) - A$, follow the descendants of this particle until they reach $L_t(s) - A - y$ for large y . Then re-incorporate them into the process.
- ▶ A particle that reaches $L_t(s) - A$ will have $M(y)$ descendants reaching $L_t(s) - A - y$, where

$$\lim_{y \rightarrow \infty} y e^{-\sqrt{2}y} M(y) = W, \quad P(W > x) \sim \frac{1}{x\sqrt{2}}.$$

Convergence to the CSBP

Let $t > 0$. For $0 \leq s \leq t$, let

$$Z_t(s) = \sum_{i=1}^{N(s)} \sqrt{2}L_t(s)e^{\sqrt{2}(X_i(s)-L_t(s))} \sin\left(\frac{\pi X_i(s)}{L_t(s)}\right) \mathbb{1}_{\{0 < X_i(s) < L_t(s)\}}.$$

The processes $(Z_t(s), 0 \leq s \leq t)$ converge as $t \rightarrow \infty$:

- ▶ Limit process has jumps of size greater than x at a rate proportional to x^{-1} .
- ▶ The jump rate at time s is also proportional to $Z_t(s)$.

Theorem (Maillard, Schweinsberg, 2020): If $Z_t(0) \Rightarrow Z$ and $L_t(0) - R(0) \rightarrow_p \infty$ as $t \rightarrow \infty$, then the finite-dimensional distributions of $(Z_t((1 - e^{-u})t), u \geq 0)$ converge as $t \rightarrow \infty$ to the finite-dimensional distributions of $(X(u), u \geq 0)$, which is a CSBP with $X(0) =_d Z$ and branching mechanism $\Psi(q) = aq + \frac{2}{3}q \log q$.

The time-change

Slightly supercritical case: particles hit L at rate $C(\log N)^{-3}$, proportional to L^{-3} .

Critical case: at time s , particles hit $L_t(s)$ at rate proportional to $L_t(s)^{-3}$, proportional to $(t - s)^{-1}$.

Using u for time in the CSBP, to get a time-homogeneous limit we need $du = (t - s)^{-1} ds$, which leads to

$$u = \log \left(\frac{t}{t - s} \right), \quad s = (1 - e^{-u})t.$$

Times near t in the BBM correspond to large times in the CSBP.

Asymptotics for the CSBP

Let $(X(u), u \geq 0)$ be a CSBP with $X(0) = x > 0$ and branching mechanism $\Psi(q) = aq + \frac{2}{3}q \log q$.

Results of Gray (1974) give

$$P_x(0 < X(u) < \infty \text{ for all } u \geq 0) = 1.$$

Letting $\alpha = e^{-3a/2}$,

$$P_x\left(\lim_{u \rightarrow \infty} X(u) = \infty\right) = 1 - e^{-\alpha x}, \quad P_x\left(\lim_{u \rightarrow \infty} X(u) = 0\right) = e^{-\alpha x}.$$

Interpretation (Bertoin, Fontbona, Martinez, 2008): The CSBP at time zero may include “prolific individuals”, whose number of descendants at time u tends to infinity as $u \rightarrow \infty$. The number of prolific individuals has a Poisson distribution with mean αx .

Survival of BBM until time t corresponds to $\lim_{u \rightarrow \infty} X(u) = \infty$.

Survival probability for BBM

Theorem (Maillard, Schweinsberg, 2020): Assume the initial configuration of particles is deterministic, but may depend on t . Recall $\zeta = \inf\{t : N(t) = 0\}$.

- ▶ If $Z_t(0) \rightarrow z$ and $L_t(0) - R(0) \rightarrow \infty$ as $t \rightarrow \infty$, then

$$\lim_{t \rightarrow \infty} P(\zeta > t) = 1 - e^{-\alpha z}.$$

- ▶ If $Z_t(0) \rightarrow 0$ and $L_t(0) - R(0) \rightarrow \infty$, then

$$P(\zeta > t) \sim \alpha Z_t(0).$$

- ▶ If at time zero there is only a single particle at x , then

$$P_x(\zeta > t) \sim \sqrt{2} \alpha \pi x e^{\sqrt{2}x - (3\pi^2 t)^{1/3}}.$$

- ▶ If at time zero there is a single particle at $L_t(0) + x$, then

$$\lim_{t \rightarrow \infty} P_{L_t(0)+x}(\zeta > t) = \phi(x),$$

where $\frac{1}{2}\phi'' = \sqrt{2}\phi' - \phi(1 - \phi)$ with $\lim_{x \rightarrow \infty} \phi(x) = 1$ and $\lim_{x \rightarrow -\infty} \phi(x) = 0$.

Asymptotics of survival time

Theorem (Maillard, Schweinsberg, 2020): Assume the initial configuration of particles is deterministic and satisfies $Z_t(0) \rightarrow 0$ and $L_t(0) - R(0) \rightarrow \infty$ as $t \rightarrow \infty$. Conditional on $\zeta > t$,

$$t^{-2/3}(\zeta - t) \Rightarrow V,$$

where V has an exponential distribution with mean $3/(\sqrt{2}c)$.

Proof: By the previous result,

$$P(\zeta > t + yt^{2/3} \mid \zeta > t) = \frac{P(\zeta > t + yt^{2/3})}{P(\zeta > t)} \sim \frac{\alpha Z_{t+yt^{2/3}}(0)}{\alpha Z_t(0)} \sim e^{-\sqrt{2}cy/3}.$$

A Yaglom-type result

For BBM at time t that will go extinct at time $t + s$:

- ▶ $Z_{t+s}(t)$ will not be close to 0 or ∞ .
- ▶ “density” of particles near y is proportional to $e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L_{t+s}(t)}\right)$.
- ▶ right-most particle is near $L_{t+s}(t) = cs^{1/3}$.
- ▶ $N(t)$ is of the order $s^{-1}e^{\sqrt{2}L_{t+s}(t)}$, so $\log N(t) \approx \sqrt{2}cs^{1/3}$.

Conditional on $\zeta > t$, the process will survive an additional $t^{2/3}V$ time units. Then $R(t) \approx c(t^{2/3}V)^{1/3} = ct^{2/9}V^{1/3}$ and $\log N(t) \approx \sqrt{2}R(t) = \sqrt{2}ct^{2/9}V^{1/3}$.

Theorem (Maillard, Schweinsberg, 2020): Assume the initial configuration of particles is deterministic and satisfies $Z_t(0) \rightarrow 0$ and $L_t(0) - R(0) \rightarrow \infty$ as $t \rightarrow \infty$. Conditional on $\zeta > t$,

$$(t^{-2/3}(\zeta - t), t^{-2/9} \log N(t), t^{-2/9} R(t)) \Rightarrow (V, \sqrt{2}cV^{1/3}, cV^{1/3}),$$

where V has an exponential distribution with mean $3/(\sqrt{2}c)$.

The conditioned BBM before time t

Theorem (Maillard, Schweinsberg, 2020): Assume the initial configuration of particles is deterministic and satisfies $Z_t(0) \rightarrow 0$ and $L_t(0) - R(0) \rightarrow \infty$ as $t \rightarrow \infty$. Conditional on $\zeta > t$, the finite-dimensional distributions of the processes

$$(Z_t((1 - e^{-u})t), u \geq 0)$$

converge as $t \rightarrow \infty$ to the finite-dimensional distributions of a CSBP with branching mechanism $\Psi(q) = aq + \frac{2}{3}q \log q$ started at 0 and conditioned to go to infinity.

Remark: The law of the CSBP started at $x > 0$ and conditioned to go to infinity has a limit as $x \rightarrow 0$. The limit can be interpreted as the process that keeps track of the number of descendants of a single prolific individual.

Populations with selection: Model 2

- ▶ The population has N individuals.
- ▶ Each individual independently acquires mutations at rate μ_N .
- ▶ Mutations are beneficial. An individual with j mutations at time t (type j individual) has fitness

$$\max\{1 + s_N(j - M(t)), 0\},$$

where $M(t)$ is average number of mutations of the N individuals at time t .

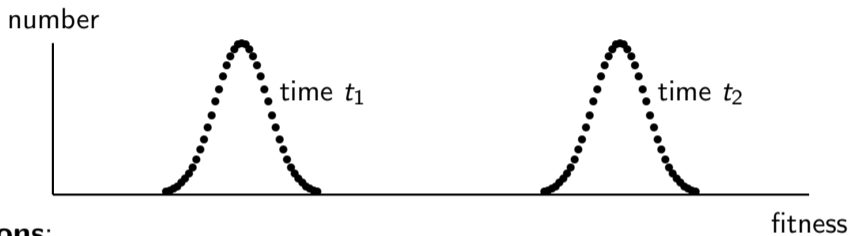
- ▶ Each individual independently lives for an exponential(1) time.
- ▶ When an individual dies, its replacement is chosen at random from the population, with probability proportional to fitness.

Other variations of the model:

- ▶ (Durrett and Mayberry, 2011): Individual with j mutations has fitness $(1 + s_N)^j$.
- ▶ (Yu, Etheridge, Cuthbertson, 2010): An individual with k mutations replaces an individual with j mutations at rate $(1 + s_N(k - j)_+)/N$.

Traveling waves

We expect the fitness distribution to evolve as a traveling wave, as proposed by Tsimring, Levine, and Kessler (1996):



Questions:

1. How fast does the fitness increase over time? (speed of traveling wave)
2. What is the fitness distribution at a given time? (shape of traveling wave)

Traveling wave believed to have a Gaussian shape: Desai and Fisher (2007), Beerenwinkel et al (2007), Rouzine, Brunet, and Wilke (2008).

Shape of traveling wave can be described using Airy function: Tsimring, Levine, and Kessler (1996), Neher and Hallatschek (2013).

One mutation at a time

If $s_N = s > 0$ and

$$\mu_N \ll \frac{1}{N \log N},$$

there will be only one beneficial mutation in population at a time that has not already spread to the entire population.

Mutations happen at rate $N\mu_N$. Then individuals with the mutation die at rate 1 and give birth at rate approximately $1 + s$, so the number of individuals with the mutation behaves like an asymmetric random walk.

With probability approximately $s/(1 + s)$, a selective sweep occurs, and the beneficial mutation spreads to the entire population.

Exponential waiting time with rate $N\mu_N s/(1 + s)$ until first selective sweep, another exponential waiting time until next one, etc.

If mutations happen faster, so more than one beneficial mutation is in the population at a time, analysis is much more complicated.

Work of Durrett and Mayberry

Durrett and Mayberry (2011) considered the case with $s_N = s > 0$ and $\mu_N \sim N^{-\beta}$, where $0 < \beta < 1$.

If T_j is the first time some individual has j mutations,

$$\frac{T_j - T_{j-1}}{\log N} \rightarrow_p t_j,$$

where $(t_j)_{j=1}^{\infty}$ is a sequence of constants depending on s and β .

Rate of adaptation is $O((\log N)^{-1})$.

Durrett and Mayberry (2011) also calculated the distribution of fitnesses at a fixed time and observed traveling wave behavior.

Only finitely many types present in the population at once.

Strong selection, faster mutation rates

Slightly faster mutations rates were studied by Desai and Fisher (2007), who described the evolution of the traveling wave and the fitness distribution.

Desai, Walczak, and Fisher (2013) found that the genealogy of the population can be described by the Bolthausen-Sznitman coalescent.

Schweinsberg (2017) studied the process rigorously.

Suppose $1/2 < c < 1$ and $0 < d < 1 - c$. Assumptions hold if:

$$\mu_N = e^{-(\log N)^c}, \quad e^{-(\log N)^d} \leq s_N \leq \frac{1}{\sqrt{\log N}}.$$

Rate of adaptation

Let

$$a_N = \frac{1}{s_N} \log \left(\frac{s_N}{\mu_N} \right), \quad k_N = \frac{\log N}{\log(s_N/\mu_N)}.$$

Theorem (Schweinsberg, 2017): Let $M(t)$ be the mean number of mutations at time t . There is a function $m : [0, \infty) \rightarrow [0, \infty)$ with

$$\lim_{t \rightarrow \infty} t^{-1} m(t) = 2$$

such that for all $t \in (0, 1) \cup (1, \infty)$, we have as $N \rightarrow \infty$,

$$\frac{1}{k_N} M(a_N t) \rightarrow_p m(t).$$

Rate at which new mutations take hold: $\frac{2k_N}{a_N} = \frac{2s_N \log N}{(\log s_N/\mu_N)^2}$.

Desai and Fisher (2007) and Rouzine, Brunet, and Wilke (2008) obtained essentially equivalent expressions by nonrigorous methods.

Fitness distribution

If $Z \sim N(\mu, \sigma^2)$, and let f be the density of Z . Then

$$\log \left(\frac{f(\mu + \ell)}{f(\mu)} \right) = -\frac{\ell^2}{2\sigma^2}.$$

Theorem (Schweinsberg, 2017): Let $X_j(t)$ be the number of individuals with j mutations at time t . There is a sequence of random times $(t_j)_{j=1}^\infty$ such that the following holds. For all $\varepsilon > 0$ and $\ell \in \mathbb{Z}$, for sufficiently large j

$$\lim_{N \rightarrow \infty} P \left(\left| \log \left(\frac{X_{j+\ell}(t_j)}{X_j(t_j)} \right) + \frac{\ell^2 [\log(s_N/\mu_N)]^2}{4 \log N} \right| > \frac{\varepsilon [\log(s_N/\mu_N)]^2}{\log N} \right) = 0.$$

Fitness distribution resembles Gaussian with variance

$\sigma_N^2 = 2 \log N / [\log(s_N/\mu_N)]^2$. Because $\sigma_N^2 \rightarrow 0$, one type dominates.

Strong selection, fast mutation rates

Yu, Etheridge, and Cuthbertson (2010) considered similar model with $s_N = s > 0$ and $\mu_N = \mu > 0$. They showed that for all $\delta > 0$, for sufficiently large N we have

$$\frac{E[M(t)]}{t} \geq (\log N)^{1-\delta}.$$

Kelly (2013) obtained a corresponding upper bound for $M(t)/t$ of order

$$\frac{\log N}{(\log \log N)^2}.$$

Weak selection, fast mutation rates

We will focus on the case when μ_N is large and s_N is small. In particular, we want to understand the shape of the traveling wave.

Related to infinitesimal model in population genetics, which goes back to Fisher (1918). See Barton, Etheridge, Véber (2017).

Fitness of an individual changes often due to mutation, like a random walk. We model this by branching Brownian motion with an inhomogeneous branching rate.

Tsimring, Levine, and Kessler (1996) and Neher and Hallatschek (2013) argued that the fitness distribution evolves like a traveling wave whose shape is given in terms of the Airy function.

Fisher (2013) and Melissa, Good, Fisher, and Desai (2021) studied intermediate cases, between those covered by Desai and Fisher (2007) and Neher and Hallatschek (2013).

Heuristic derivation of traveling wave shape

Tsimring, Levine, and Kessler (1996) and Neher and Hallatschek (2013) derived shape of traveling wave.

Suppose $q(x, t)$ denotes density of particles with fitness x at time t . Let $m(t)$ be the mean fitness at time t . Then

$$\frac{\partial}{\partial t} q(x, t) = (x - m(t))q(x, t) + \mu(q(x - s, t) - q(x, t)). \quad (1)$$

Assume $m(t) = vt$ and $q(x, t) = f(x - vt)$. Write $z = x - vt$ for relative fitness.

Now

$$\frac{\partial}{\partial t} q(x, t) = \frac{\partial}{\partial t} f(x - vt) = -vf'(x - vt) = -vf'(z),$$

so (1) becomes

$$-vf'(z) = zf(z) + \mu(f(z - s) - f(z)).$$

Now use the approximation $f(z - s) - f(z) \approx -sf'(z) + \frac{1}{2}s^2f''(z)$ to get

$$-vf'(z) = zf(z) - \mu sf'(z) + \frac{1}{2}\mu s^2 f''(z).$$

The Airy shape

We have
$$-vf'(z) = zf(z) - \mu sf'(z) + \frac{1}{2}\mu s^2 f''(z).$$

Write $\sigma^2 = v - \mu s$ and $D = \frac{1}{2}\mu s^2$. Then

$$zf(z) + \sigma^2 f'(z) + Df''(z) = 0.$$

Let $g(z) = e^{\sigma^2 z/2D} f(z)$. Then

$$g''(z) = -g(z) \left(\frac{z}{D} - \frac{\sigma^4}{4D^2} \right).$$

The Airy function satisfies $Ai''(z) = zAi(z)$. If $g(z) = Ai(a + bz)$, then $g''(z) = b^2(a + bz)g(z)$. Setting

$$b = -\frac{1}{D^{1/3}}, \quad a = \frac{\sigma^4}{4D^{4/3}}$$

leads to

$$f(z) = e^{-\sigma^2 z/2D} Ai \left(\frac{\sigma^4}{4D^{4/3}} - \frac{z}{D^{1/3}} \right).$$

Modeling the evolution of populations undergoing selection using branching Brownian motion

Lecture V

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Populations with selection: Model 2

- ▶ The population has N individuals.
- ▶ Each individual independently acquires mutations at rate μ_N .
- ▶ Mutations are beneficial. An individual with j mutations at time t has fitness

$$\max\{1 + s_N(j - M(t)), 0\},$$

where $M(t)$ is average number of mutations of the N individuals at time t .

- ▶ Each individual independently lives for an exponential(1) time.
- ▶ When an individual dies, its replacement is chosen at random from the population, with probability proportional to fitness.

We aim to understand rigorously the case when μ_N is large and s_N is small.

Fitness of an individual changes often due to mutation, like a random walk.

We approximate this by branching Brownian motion. Because fitter particles have more offspring, we will consider BBM with an inhomogeneous branching rate.

BBM with an inhomogeneous branching rate

Begin with some configuration of particles on \mathbb{R} at time zero.

The position of the particle represents the fitness of the individual.

Particles independently move as one-dimensional Brownian motion with drift $-\rho$.

Each particle dies at rate $d(x) = 1$.

A particle at x splits into two at rate $b(x) = 1 + \beta x$.

For $x < -1/\beta$, set $b(x) = 0$, $d(x) = -\beta x$, so $b(x) - d(x) = \beta x$.

As long as $b(x) - d(x) = \beta x$, our results hold if there exists $C > 0$ such that $d(x) \geq C$ for all $x \in \mathbb{R}$ and $b(x) \leq 1/C$ for all $x \leq 1/\beta$. Also, some of our results require b to be nondecreasing and d to be nonincreasing.

Choose the parameters ρ and β and the initial configuration of particles so that the number of particles remains roughly stable over time (to model a population of approximately fixed size).

Results on this model based on joint work with Matt Roberts and Jiaqi Liu.

Related work

Model was studied by Neher and Hallatschek (2013).

Beckman (2019) considered BBM with an inhomogeneous branching rate, focused on a shorter time scale.

BBM with a periodic branching rate studied by: Hamel, Nolen, Roquejoffre and Ryzhik (2016), Lubetzky, Thornett, and Zeitouni (2018), and Ren, Song, and Yang (2022).

Harris and Harris (2009) considered BBM with no drift, no deaths, and branching at rate $b(x) = \beta|x|^p$. Let $R(t)$ be the position of right-most particle at time t .

1. If $0 < p < 2$, then $t^{-2/(2-p)}R(t) \rightarrow c$ a.s.
2. If $p = 2$, then $t^{-1} \log R(t) \rightarrow \sqrt{2\beta}$ a.s.
3. If $p > 2$, the process explodes in finite time.

Berestycki, Brunet, Harris, Harris, Roberts (2015) obtained more detailed results about the number of particles in different regions using large deviations methods.

Large deviations (Berestycki, Brunet, Harris, Harris, and Roberts (2015))

Schilder's Theorem: Let $f : [0, T] \rightarrow \mathbb{R}$. The probability that Brownian motion with drift $-\rho$, started from $f(0)$, stays “close” to f until time T is approximately

$$\exp\left(-\frac{1}{2}\int_0^T (f'(u) + \rho)^2 du\right).$$

Many-to-one Lemma: Recall $b(x) - d(x) = \beta x$. The expected number of particles in BBM at time T for which trajectory of the ancestor stays close to f :

$$\exp\left(\int_0^T \beta f(u) - \frac{1}{2}(f'(u) + \rho)^2 du\right).$$

The actual number of particles that stay close to f will be comparable to the expectation, if the integral up to t is nonnegative for all $t \in [0, T]$.

If $f(u) = \rho^2/2\beta$ for all $u \in [0, T]$, then the integrand is zero. If we start with one particle near $\rho^2/2\beta$, right-most particle stays near $\rho^2/2\beta$, if the population does not quickly die out.

The empirical distribution of particles

Start with one particle at $\rho^2/2\beta$. The number of particles near y at time T is approximately

$$\exp\left(\int_0^T \beta f_y(u) - \frac{1}{2}(f_y'(u) + \rho)^2 du\right),$$

where the function f_y maximizes the integral subject to the conditions $f_y(0) = \rho^2/2\beta$, $f_y(T) = y$, and that the integral up to t is nonnegative for all $t \in [0, T]$.

For sufficiently large T , the number of particles near y is approximately $\exp(g(y))$, where

$$g(y) = \frac{\rho^3}{2\beta} - \rho y - \frac{2\sqrt{2\beta}}{3} \left(\frac{\rho^2}{2\beta} - y\right)^{3/2}, \quad y \leq \frac{\rho^2}{2\beta}.$$

Analyzing the function g

The number of particles near y is approximately $\exp(g(y))$,

$$g(y) = \frac{\rho^3}{2\beta} - \rho y - \frac{2\sqrt{2\beta}}{3} \left(\frac{\rho^2}{2\beta} - y \right)^{3/2}.$$

Because $g'(0) = 0$, $g''(0) = -\beta/\rho$, the distribution of particles at time T is approximately Gaussian with mean 0 and variance ρ/β .

Asymptotics for the Airy function imply

$$\exp(g(y)) \approx \exp\left(\frac{\rho^3}{2\beta} - \rho y\right) \text{Ai}\left((2\beta)^{1/3} \left(\frac{\rho^2}{2\beta} - y\right)\right).$$

Shape is described by the Airy function as predicted by Tsimring, Levine, and Kessler (1996) and Neher and Hallatschek (2013).

Since $g(\rho^2/2\beta) = g(-5\rho^2/8\beta) = 0$, right-most particle should be near $\rho^2/2\beta$ and left-most particle should be near $-5\rho^2/8\beta$.

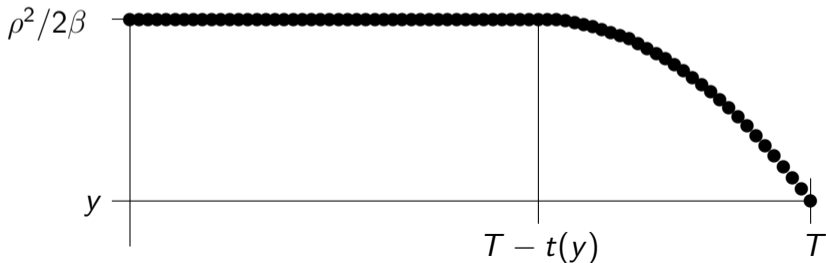
The trajectory of particles

Let

$$t(y) = \sqrt{\frac{2}{\beta} \left(\frac{\rho^2}{2\beta} - y \right)}.$$

Particles near y at time T typically follow the trajectory f_y :

$$f_y(t) = \begin{cases} \frac{\rho^2}{2\beta} & \text{if } 0 \leq t \leq T - t(y) \\ \frac{\rho^2}{2\beta} - \frac{\beta}{2} (t - (T - t(y)))^2 & \text{if } T - t(y) \leq t \leq T \end{cases}$$



Assumptions on the parameters

For large deviations heuristics to be accurate, we need:

1. The standard deviation of the Gaussian $\sqrt{\rho/\beta}$ should be much smaller than $\rho^2/2\beta$. That is, $\rho^3 \gg \beta$.
2. βx should be small when $x = \rho^2/2\beta$. That is, $\rho \ll 1$.

Truncating the process at L

Let $q_t^L(x, y)$ be the density when particles are killed at L , meaning that if $A \subset (-\infty, L)$, the expected number of particles in A at time t is

$$\int_A q_t^L(x, y) dy.$$

We have

$$\frac{\partial}{\partial t} q_t^L(x, y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} q_t^L(x, y) + \rho \frac{\partial}{\partial y} q_t^L(x, y) + \beta y q_t^L(x, y).$$

Looking for solutions of the form $e^{\lambda t} h(y)$, we have

$$\lambda h(y) = \frac{1}{2} h''(y) + \rho h'(y) + \beta y h(y), \quad h(L) = 0, \quad \lim_{y \rightarrow -\infty} h(y) = 0.$$

Solutions are

$$h_n(y) = e^{-\rho y} \varphi_n(y), \quad \varphi_n(y) = \text{Ai}((2\beta)^{1/3}(L - y) + \gamma_n),$$

where $\dots < \gamma_2 < \gamma_1 < 0$ denote the zeros of the Airy function.

Deriving the density formula

We have

$$q_t^L(x, y) = \sum_{n=1}^{\infty} a_n e^{-\rho y} \varphi_n(y) = \sum_{n=1}^{\infty} a_n e^{-\rho y} \text{Ai}((2\beta)^{1/3}(L - y) + \gamma_n).$$

Choose the constants a_n so that the density converges to a unit mass at x as $t \rightarrow 0$. This leads to (Salminen, 1988):

$$q_t^L(x, y) = (2\beta)^{1/3} \sum_{n=1}^{\infty} \frac{e^{(\beta(L+(2\beta)^{-1/3}\gamma_n) - \rho^2/2)t}}{(Ai'(\gamma_n))^2} e^{\rho x} \varphi_n(x) e^{-\rho y} \varphi_n(y).$$

Choose

$$L = \frac{\rho^2}{2\beta} - (2\beta)^{-1/3} \gamma_1.$$

If t is large enough that the $n = 1$ term dominates, then

$$q_t^L(x, y) \approx \frac{(2\beta)^{1/3}}{(Ai'(\gamma_1))^2} e^{\rho x} \varphi_1(x) e^{-\rho y} \varphi_1(y).$$

Observations related to density formula

$$q_t^L(x, y) \approx \frac{(2\beta)^{1/3}}{(Ai'(\gamma_1))^2} e^{\rho x} \varphi_1(x) e^{-\rho y} \varphi_1(y).$$

Let $N(t)$ be the number of particles at time t , and denote by $X_1(t) \geq X_2(t) \geq \dots \geq X_{N(t)}(t)$ the positions of particles.

1. Let

$$Z(t) = \sum_{i=1}^{N(t)} e^{\rho X_i(t)} Ai((2\beta)^{1/3}(L - X_i(t)) + \gamma_1) \mathbb{1}_{\{X_i(t) < L\}}.$$

When particles at L are killed, $(Z(t), t \geq 0)$ is a martingale. We can use $Z(t)$ to measure “size” of the process at time t . Particles within $O(\beta^{-1/3})$ of L contribute most to $Z(t)$.

2. The density of particles near y is roughly proportional to

$$e^{-\rho y} Ai((2\beta)^{1/3}(L - y) + \gamma_1).$$

Strategy for studying configuration of particles

To study the particles within $O(\beta^{-1/3})$ of L , consider the process in which particles are killed at L :

- ▶ Show that particles do not reach L frequently.
- ▶ Use first moment calculations to show that the “density” of particles near y is proportional to $e^{-\rho y} Ai((2\beta)^{1/3}(L - y) + \gamma_1)$.
- ▶ Use second moment bounds to control fluctuations, show configuration of particles near L is relatively stable over time.

To study the configuration of particles near y at a large time T :

- ▶ Observe that particles near y at time T are descended from ancestors within $O(\beta^{-1/3})$ of L at time $T - t(y)$. These ancestors are in a stable configuration as noted above.
- ▶ Use first and second moment estimates to show the number of particles near y is highly concentrated around its mean.

Assumptions on parameters and initial conditions

Consider a sequence of branching Brownian motions indexed by n . Suppose

$$\lim_{n \rightarrow \infty} \frac{\rho_n^3}{\beta_n} = \infty, \quad \lim_{n \rightarrow \infty} \rho_n = 0. \quad (2)$$

Assume the initial configuration is such that

$$Z_n(0) \asymp_p \frac{\beta_n^{1/3}}{\rho_n^3} e^{\rho_n L_n}, \quad (3)$$

where \asymp_p means the ratio of the two sides is tight, and

$$Y_n(t) = \sum_{i=1}^{N_n(t)} e^{\rho_n X_{i,n}(t)}, \quad \rho_n^2 e^{-\rho_n L_n} Y_n(0) \rightarrow_p 0. \quad (4)$$

Gaussian shape for particle configuration

Theorem (Roberts and Schweinsberg, 2021): Suppose assumptions (1), (2), and (3) hold. For $t \geq 0$, define the random probability measure

$$\zeta_n(t) = \frac{1}{N_n(t)} \sum_{i=1}^{N_n(t)} \delta_{X_{i,n}(t)\sqrt{\beta_n/\rho_n}}.$$

Let $1 < a < \infty$, and let $t_n = a\rho_n/\beta_n$. Let μ be the standard normal distribution. Then, as $n \rightarrow \infty$,

$$\zeta_n(t_n) \Rightarrow \mu.$$

Interpretation of drift

Empirical distribution of particle locations: normal with mean 0 and variance ρ/β .

Our model is equivalent to one with no drift and, at time t ,

$$b(x) - d(x) = \beta(x - \rho t).$$

Then the empirical distribution of particles after a long time t will be approximately normal with mean ρt , so ρ is the speed at which the Gaussian traveling wave advances.

Fisher's Fundamental Theorem of Natural Selection (1930): The rate at which fitness of a population increases is the variance of the fitness distribution.

1. Rate at which fitness increases: $\beta\rho$.
2. Variance of fitness distribution: $\beta^2(\rho/\beta) = \beta\rho$.

Note: Number of particles in the long-run is

$$N \asymp \frac{\beta^{1/3}}{\rho^3} \exp\left(\frac{\rho^3}{6\beta} - \rho(2\beta)^{-1/3}\gamma_1\right),$$

Particle configurations near right edge

Most particles within $O(\sqrt{\rho/\beta})$ of the origin.

Next result considers particles within $O(\beta^{-1/3})$ of L , which are the particles that will have descendants alive far into the future.

Theorem (Roberts and Schweinsberg, 2021): Suppose assumptions (1), (2), and (3) hold. For $t \geq 0$, define the random probability measure

$$\xi_n(t) = \frac{1}{Y_n(t)} \sum_{i=1}^{N_n(t)} e^{\rho_n X_{i,n}(t)} \delta_{(2\beta_n)^{1/3}(L_n - X_{i,n}(t))}.$$

Let $0 < a < \infty$, and let $t_n = a\rho_n/\beta_n$. Then $\xi_n(t_n) \Rightarrow \nu$, where ν is the probability measure on $(0, \infty)$ with probability density function

$$h(y) = \frac{\text{Ai}(y + \gamma_1)}{\int_0^\infty \text{Ai}(z + \gamma_1) dz}.$$

The full shape of the traveling wave

Let $L_n^* = \rho_n^2/2\beta_n$ and $L_n^\dagger = -5\rho_n^2/8\beta_n$.

Let $N_t(A)$ be the number of particles in A at time t .

Theorem (Liu and Schweinsberg, 2022): Suppose assumptions (1), (2), and (3) hold. Consider a sequence of intervals $(I_n)_{n=1}^\infty = ([a_n, b_n])_{n=1}^\infty$ such that

$$b_n - a_n \gg 1, \quad L_n^* - a_n \gg \beta_n^{-1/3}, \quad b_n - L_n^\dagger \gg \beta_n^{-1/3}.$$

Let z_n be the point in I_n closest to the origin. Recall that

$$t_n(y) = \sqrt{\frac{2}{\beta_n} \left(\frac{\rho_n^2}{2\beta_n} - y \right)}.$$

Suppose $t_n - t_n(z_n) \asymp \rho_n/\beta_n$. Let $I_n^* = I_n \cap (-\infty, L_n^*)$. Then

$$N_{t_n}(I_n) \sim_p \frac{Z_n(t_n - t_n(z_n))e^{-\rho_n^3/2\beta_n}}{Ai'(\gamma_1)^2} \int_{I_n^*} \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y)} dy$$

where \sim_p means the ratio of the sides tends to 1 in probability.

The left-most and right-most particles

Let $R_n(t)$ be the position of the right-most particle at time t . Let $G_n(t)$ be the position of the left-most particle at time t . Let

$$L_n = L_n^* - (2\beta_n)^{-1/3}\gamma_1, \quad \bar{L}_n = L_n^\dagger + 2(2\beta_n)^{-1/3}\gamma_1.$$

Theorem (Liu and Schweinsberg, 2022): Suppose assumptions (1), (2), and (3) hold. Let $0 < a < \infty$, and let $t_n = a\rho_n/\beta_n$. For any $C_1 > 0$ and $C_2 > 0$,

$$\lim_{n \rightarrow \infty} P\left(L_n - \frac{C_1}{\beta_n^{1/3}} \leq R_n(t_n) \leq L_n + \frac{C_2}{\rho_n}\right) = 1.$$

If $a > 3/2$, then for all $\varepsilon > 0$, there exist $C_3 > 0$ and $C_4 > 0$ such that for sufficiently large n ,

$$P\left(\bar{L}_n - \frac{C_3}{\rho_n} \leq G_n(t_n) \leq \bar{L}_n + \frac{C_4}{\beta_n^{1/3}}\right) > 1 - \varepsilon.$$

BBM with critical branching

Consider a Bienaymé-Galton-Watson process $(Z_n)_{n=0}^{\infty}$ whose offspring distribution has mean $1 + \varepsilon$.

- ▶ If $Z_0 = 1$, the probability that the process survives forever is $O(\varepsilon)$.
- ▶ If $Z_0 \gg \varepsilon^{-1}$, the process evolves like a supercritical branching process.
- ▶ If $Z_0 \ll \varepsilon^{-1}$, the process evolves like a critical branching process.

For a particle at L_n , we have $b(L_n) - d(L_n) = \beta_n L_n \approx \rho_n^2/2$.

If $t_n = O(\rho_n/\beta_n)$, number of particles reaching L_n by time t_n will be $O(1/\rho_n^2)$.

Theorem (Sawyer and Fleischman, 1979): Consider BBM started with one particle at 0 in which each particle dies at rate 1 and splits into two particles at rate 1. Let M be the maximum position that any particle ever attains. Then

$$P(M > x) \sim 6/x^2 \quad \text{as } x \rightarrow \infty.$$

We use this result to show that, with high probability, no particle alive at time t_n is more than $o(1/\rho_n)$ above L_n .

Genealogy of the population

Neher and Hallatschek (2013) argued nonrigorously that the genealogy of the population can be described by the Bolthausen-Sznitman coalescent.

We believe that this result follows from the same ideas used for the case of BBM with absorption by Berestycki, Berestycki, and Schweinsberg (2013).

Particles that reach L may produce enough offspring to cause multiple mergers of ancestral lines.

Particles reach L at rate β/ρ^3 . Because $b(L) - d(L) \approx \rho^2/2$, a particle that reaches L has probability approximately $\rho^2/2$ to have many surviving offspring. Therefore, the rate of events in which we see large jumps in the number of particles is $O(\beta/\rho)$.

Multiple mergers occur on the time scale ρ/β .

Returning to the original population model

In one time unit, an individual gets a $\text{Poisson}(\mu)$ number of mutations, which increase fitness by s . The variance of the change in fitness is $s^2\mu$. In BBM, the variance of the change in fitness in one time unit is β^2 , so

$$\beta = s\sqrt{\mu}.$$

Also,

$$N \asymp \frac{\beta^{1/3}}{\rho^3} \exp\left(\frac{\rho^3}{6\beta} - \rho(2\beta)^{-1/3}\gamma_1\right),$$

which leads to $\rho^3 \sim 2\beta \log(N^3\beta^2)$. Therefore the condition $\rho^3/\beta \rightarrow \infty$ is equivalent to

$$N^3 s^2 \mu \rightarrow \infty.$$

Random walk should be well approximated by Brownian motion when $\mu \gg \rho^2$, or

$$\frac{\mu}{s} \gg \log(N^3 s^2 \mu).$$

These conditions appeared in Melissa, Good, Fisher, and Desai (2021).

Further implications for population models

When these conditions hold, we conjecture that results for BBM with an inhomogeneous branching rate carry over to the original population model.

Results for BBM should carry over to populations with deleterious mutations, or random fitness effects.

Result for BBM that fitness of the population increases at rate $\beta\rho$ could imply:

- ▶ **Rate of adaptation:** beneficial mutations
- ▶ **Muller's ratchet:** deleterious mutations
- ▶ **Mutation-selection balance:** both beneficial and deleterious mutations

Could be fruitful area for future work. Technical challenges:

- ▶ Random walk rather than Brownian motion.
- ▶ Fixed population size N gives less independence.
- ▶ Different behavior: jumps in the population size in the BBM model lead to many nearly simultaneous deaths in a population of fixed size.

Summary of Lectures

For many classical population models, genealogy given by Kingman's coalescent.

For populations with large family sizes, the genealogy may be described by a coalescent with multiple mergers. The Beta($2 - \alpha, \alpha$) coalescent describes the genealogy when individuals have k or more offspring with probability $Ck^{-\alpha}$.

We introduced two population models involving selection.

- ▶ Model 1: Individuals are killed when their fitness gets too low.
Can be approximated by BBM with absorption.
- ▶ Model 2: Individuals with higher fitness have more offspring.
Can be approximated by BBM with an inhomogeneous branching rate.

In both models, we can understand the fitness distribution of the population.

- ▶ Model 1: Most particles are located near the barrier at the origin.
- ▶ Model 2: The bulk of the fitness distribution is Gaussian. The tails can be described better using the Airy function.

Summary of genealogies

The Bolthausen-Sznitman coalescent often describes the genealogy of populations undergoing selection.

Model 1 (BBM with absorption):

- ▶ Obtained nonrigorously by Brunet, Derrida, Mueller, Munier (2006, 2007).
- ▶ Proved rigorously by Berestycki, Berestycki, and Schweinsberg (2013).

Model 2 (BBM with inhomogeneous branching rate):

- ▶ Strong selection: obtained nonrigorously by Desai, Walczak, and Fisher (2013) and rigorously by Schweinsberg (2017).
- ▶ Weak selection, fast mutation rates: obtained nonrigorously by Neher and Hallatschek (2013). Should follow rigorously from ideas in Roberts and Schweinsberg (2021).

Beta coalescents may also appear, especially with semi-pushed waves (Birzu, Hallatschek, Korolev, 2018; Cortines and Mallein, 2018; Tourniaire, 2021).