Homework #1

(due Wednesday, October 4, in class)

This assignment is based on the material covered in the lecture on Friday, September 27. Durrett's book covers this material only briefly at the beginning of section 1.1. If you want some additional reading that includes more examples, see sections 1.5-1.7 of Resnick, *A Probability Path*.

1. (Durrett, Exercise 1.1.4 (i), p. 8) Suppose $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are σ-fields of subsets of Ω such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$. Prove that
$$\bigcup_{n=1}^{\infty} \mathcal{F}_n$$
is a field.

2. Let Ω be the set of all positive integers. Let $\mathcal{F}$ be the collection of all sets $A \subseteq \Omega$ such that either A or A^c is finite.

   (a) Is $\mathcal{F}$ a field? Prove that your answer is correct.

   (b) Is $\mathcal{F}$ a σ-field? Prove that your answer is correct.

3. (a) Let $\Omega = \{1, 2, 3, 4\}$. Let $\mathcal{A} = \{\{1\}, \{1, 2, 3\}\}$. Find $\sigma(\mathcal{A})$, and show that your answer is correct.

   (b) Let Ω be the set of all positive integers, and let $\mathcal{A}$ be the collection of one-element subsets of Ω. Find $\sigma(\mathcal{A})$, and show that your answer is correct.

4. Let $\Omega = \mathbb{R}$, and let $\mathcal{A}$ be the collection of all subsets of Ω of the form
$$\bigcup_{i=1}^{k} (a_i, b_i]$$
with $k \geq 0$ and $-\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq \infty$. (We interpret the union to be $\emptyset$ if $k = 0$, and we interpret $(a_i, b_i]$ to be (a_i, ∞) if $b_i = \infty$.) Show that $\mathcal{A}$ is a field.

5. Let $\mathcal{A}$ be the field defined in Problem 4. Show that $\sigma(\mathcal{A})$ is the σ-field $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$.

   Note: Recall that we defined $\mathcal{B}(\mathbb{R})$ to be the σ-field generated by the open subsets of $\mathbb{R}$. You should work from this definition, rather than the alternative (but equivalent) definition given in Resnick's book.