An Approach to Fractions Via Measurement

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Introduction

No matter how many diagrams they see of pizzas divided into slices, students have persistent difficulties with understanding fractions. What does a fraction mean? Where do the rules for fraction arithmetic come from? What does multiplication, or division, of fractions represent? What is the role of "proportional reasoning" in all this? What is the significance of the rational number system which is finally constructed? How can there be irrational numbers, and what is the significance of this?

The approach presented in this paper has two related sources. One is measurement: fractions arise when we realize that simply counting objects is inadequate for practical needs; we need to measure objects as well. Although $3 > 1$, three slices of a medium pizza may or may not be more filling than one small pizza. Whether we measure lengths, angles, areas, volumes, or masses, the underlying reasoning is the same and leads directly to the concept of fractions. This reasoning is fundamental to physics, chemistry, and the other sciences, and is an instance of proportional reasoning. The second source is Greek geometry, as summarized in Euclid's *Elements*, for example. Although Euclid had no rational numbers in the modern sense, and never assigned a numerical measure to any geometric quantity, he was adept at comparing segments, angles, and areas. He could divide a segment into any number of equal parts and determine the greatest common unit measuring two segments. He recognized the importance of commensurability, namely whether a pair of quantities can each be expressed as a whole number of common units, and, following Eudoxus, overcame the crisis of realizing that this was not always possible.

Measurement

Although measurements of length, angle, area, and so forth all depend on the same principles, in this paper I will focus on measuring the length of line segments for definiteness. To measure a segment, we need to decide on a unit of length (a segment whose length is agreed to be one unit), and we must be able to count units, add them (that is, lay them end to end, forming a longer segment), and compare segments (that is, determine whether they are congruent or one is longer than the other). If n unit segments laid end to end are congruent to the segment to be measured, we assign it a length of n. If not, we can at least make a statement of the form, "The segment has length between n and $n+1$." The problem is then to assign a specific length to such a segment. (Measurement of areas or volumes is conceptually harder than that of segments because the wide variety of possible shapes makes it hard to decide whether two volumes, say, are equal. One can imagine that the volumes are transparent containers of fluid, and pour the fluid from one into the other to compare them.)

Suppose for the moment that a segment is congruent to a chain of n units, therefore having length n. If we divide each unit into two equal parts, and use these as new measuring units, the same segment is congruent to $2n$ new units, so its measured length has doubled. This simple but crucial observation is how proportional reasoning applies to measurement. *The length of any segment measured in smaller [larger] units will be proportionally larger [smaller].* Since inches are twelve times smaller than feet, any length will contain twelve times as many inches as feet. Since centimeters are 2.54 times smaller than inches, any length will contain 2.54 times more centimeters than inches. It is critical that students internalize such reasoning, rather than memorizing formulas for converting units, in order to understand fractions and measurement.

If a segment does not have an integer length, we can attempt to measure it by cutting our measuring unit into smaller parts and using them as finer units. If we cut the unit into b equal parts, and the segment is congruent to a of these smaller units, we assign the symbol a/b to represent its length. For example, the smaller units themselves will have length $1/b$.

It is good to recognize that dividing a unit into several exactly equal parts is easier said than done. Euclid does it for segments by constructions using the theory of parallel lines and similar triangles. However, if we are measuring angles, a construction for trisecting an arbitrary unit angle is famously impossible. Dividing areas or volumes into equal parts is also challenging. For practical purposes one can use trial and error: guess the part, add the several equal parts together, and readjust the part if the sum is larger or smaller than one unit.

There is an alternative way of measuring that avoids the problem of subdividing the unit. We can use multiple copies of the segment to be measured as well as the unit. If b copies of the segment (laid end to end) are congruent to a units, we again declare the segment to have length a/b. (See Figure 1.) How do we know that this new procedure always yields the same result as the original one? Well, take a segment having length a/b according to the original procedure: it is congruent to a of the smaller units $1/b$. Then b copies of the segment will be congruent to ab smaller units, which is a full-size units. Therefore the new procedure will lead to the same length a/b. Conversely, take a segment having length a/b according to the new procedure: b copies of it are congruent to a units, which is ab smaller units. Then a single copy of the segment is congruent to a smaller units, and therefore the original procedure leads to the same length a/b.

In the present approach to fractions it is crucial that students justify claims such as the equivalence of the two methods of measuring length by fully articulating these chains of proportional reasoning. Every claim we will make about fractions can be understood in this way.

To justify thinking of the symbols a/b as *numbers*, we must explain how to do arithmetic with them: how they are compared in size, added, subtracted, multiplied, and divided. Each of these operations will be assigned a clear meaning in terms of measurement, from which the rules can be derived by proportional reasoning.

First, observe that a segment having integer length n will also be assigned length $n/1$ by our definitions. So we say $n/1 = n$, meaning simply that these denote the length of the same object. If b copies of a segment equals a units, then kb copies of the segment will equal ka units for any natural number k. Therefore we must say that $a/b = ka/kb$ for all natural numbers a, b, k. Thus $2/3 = 4/6 = 6/9 = \cdots$. Observe that we have made no reference to "cancellation" or common factors, simply proportional reasoning. Similarly, $20/4 = 4(5)/4(1)$ is an integer, namely 5. In general, whenever a is a multiple of b, a/b agrees with the quotient $a \div b$. In this sense, fractions generalize division of natural numbers to cases where the quotient was previously undefined, or expressed awkwardly in terms of a remainder. When we define arithmetic operations, we will do so in terms of lengths, which ensures that any two "equivalent fractions" are indeed interchangeable in arithmetic. The fact that fractions have infinitely many equivalent representations is a major conceptual obstacle for students. It is also an opportunity, since one can choose the representation best suited to any problem. The approach via proportional reasoning should at least make it crystal clear why the equivalence occurs.

Given fractions a/b and c/d, which is larger? b copies of the first is a units, d copies of the second is c units. To compare, we must find a common multiple yielding an integral number of units for both. For example, bd copies of the first is ad units, bd copies of the second is bc. Therefore the fractions can be compared (equal, greater, or less) by comparing the integers ad and bc accordingly. This is the usual cross-multiplication rule, derived from proportional reasoning. Students should practice this reasoning in specific numerical examples before applying it to the symbolic fractions a/b and c/d. Another way to explain it is that we have measured both fractions using the much smaller common unit $1/bd$, in terms of which both have integral measure. Euclid stressed the importance of such "commensurable" magnitudes, which become integral in terms of a smaller unit. A basic principle of the theory of fractions is that any two are commensurable.

At this point we can raise a crucial philosophical issue. We have now assigned lengths to many line segments, but does that mean all of them? Do there exist line segments which still have not been assigned any length? A segment which has escaped our measuring process would have to have a most peculiar property: regardless of how many parts, b, we divide our measuring unit into, no whole number a of such parts matches the segment! The segment is *incommensurable* with our chosen unit. I suggest polling students as to whether this is possible. Without further mathematical knowledge, I suspect my own intuition would come down against it. Surely it would create a crisis for our developing theory of measurement, as it did for the Greeks. To fully appreciate irrational numbers and the structure of the real number line, students must ponder this issue and appreciate its depth. It is not at all clear how it can be definitively resolved. We return to it after completing the development of rational number arithmetic.

Arithmetic

To add fractions, we must reflect on what addition is supposed to mean. The sum of two integral lengths is simply the length of the combined segment, so we keep the same meaning for addition of fractions. What is the length of the segment obtained by combining (laying end to end) segments a/b and c/d? We already know that in terms of the smaller unit $1/bd$, the answer is $ad + bc$. Thus, by definition, $a/b + c/d = (ad + bc)/bd$. The smaller unit is what is usually called the "common denominator", but hopefully now with more meaning attached.

Subtraction is similar. Assuming that $a/b > c/d$, we can subtract the shorter segment from the longer, meaning that we cut the longer segment into two parts, one of which is the shorter segment, and the other is called their difference. This definition clearly makes addition and subtraction inverse operations. We obtain $a/b - c/d = (ad - bc)/bd$ by the same reasoning applied to addition.

Given an "improper fraction", meaning a/b where $a \geq b$, we can convert it to a "mixed number" as follows. Such a segment is congruent to a small units $1/b$. Separate off b of these, forming a segment of length $b/b = 1$ with the length $(a - b)/b$ remaining. We have shown that $a/b = 1 + (a - b)/b$. By repeating this operation we can decompose a/b into a natural number n, chosen as large as possible, and a proper fraction remainder r/b (possibly zero). These numbers are familiar from elementary school, where we would say that $a \div b$ equals n, with a remainder of r.

Multiplication of fractions is confusing because students often lack a model for what it could mean. It is said that repeated addition is not a suitable model: I know what it means to add four 5's, but how do I add four and a half 5's? (I say "add four 5's" rather than the common alternative "add 5 to itself four times", because the latter should literally mean $5+5+5+5+5$, having four plus signs rather than four fives!) Actually this is not hard to interpret: it must mean, add four 5's and then a segment half the length of 5. Nevertheless, we will adopt a model more closely linked to measurement. $3 \times 2 = 6$ can be interpreted as: three segments of length 2 make a segment of length 6. That is, if we adopt a segment of length 2 as a new unit, and measure out 3 of these units, we actually obtain a segment of length 6. We use the same model for the meaning of $(a/b) \times (c/d)$: if we adopt a segment U of length c/d as a new unit, and another segment S measures a/b *of these units*, how long is S "really"? Note that this model incorporates the rule, "of" means multiply: we are taking a/b *of* the unit. (Of course, it is the context rather than the simple occurrence of the word "of" which indicates multiplication. One can easily create a problem in which "of" appears which is not solved by multiplication.) According to the definition of a/b, b copies of S equals a copies of U. According to the definition of c/d, d copies of U equals c. Therefore bd copies of S equals ad copies of U, which is ac, so that the length of S is ac/bd. This confirms the usual rule for multiplying fractions and shows that multiplication is commutative although our definition does not treat a/b and c/d symmetrically. Again students should apply this reasoning to numerical examples before relying on formulas.

Division of fractions again requires a suitable model, or an agreement that it should be the inverse operation to multiplication. Our model for $6 \div 2 = 3$ will be: if a length of 6 is measured using units of length 2, the measured "length" is 3 of these units. In general, $a/b \div c/d$ is the measurement of a length a/b when the unit is chosen to be c/d. Now use proportional reasoning: the unit c/d is d times smaller than c, so the measurement will be d times larger than if c were used as unit: $a/b \div c/d = d(a/b \div c)$. In turn, $a/b \div c$ is c times smaller than $a/b \div 1$, hence it is a/bc. Finally, $a/b \div c/d = d(a/bc) = ad/bc$. The result is consistent with the usual "invert and multiply" rule, but the reasoning makes it clear exactly how the factor d escaped from the denominator of the unit and wound up in the numerator of the result.

Our models of multiplication and division of fractions are flexible enough to apply to situations which at first glance do not seem to involve measurement. For example, we have many identical boxes, each containing the same number of objects, or the same quantity of material: a dozen eggs, 12/7 pounds of concrete, or whatever. How many objects, or what quantity of material, is contained in a given number of boxes: five boxes, three and a quarter, etc.? Conversely, how many boxes will it take to hold a given total quantity? If we take an individual object, or a pound of concrete, as our measuring unit, then a box can be viewed as an alternative unit. The situation fits our models for multiplication or division: the quantity of concrete in $3\frac{1}{4}$ boxes is $3\frac{1}{4} \times 12/7 = 39/7$ pounds. The number of boxes that contains 40 eggs is $40 \div 12 = 10/3$ boxes. Students still have to think about what this means in situations where the "units" are not divisible: 10/3 boxes actually means four boxes, of which the fourth is only partially filled.

There is a common but subtle pedagogical error in which the teacher tries to explain why, for example, $3x + 5x = 8x$ is correct. Rather than invoke the distributive property, an analogy is drawn between x and an object or unit: "Three apples plus five apples equals eight apples; three meters plus five meters equals eight meters." However, the analogy is inadequate because x is a quantity having a numerical value, not an object or unit; $3x$ means multiplication of the value of x by three, not simply counting out three objects. This "units model of variables" error can be corrected while preserving the intuition behind it by viewing x as the number of objects, or quantity of material, in a box. "Three boxes plus five boxes equals eight boxes" is true in the sense that the quantity of material in three boxes, plus the quantity in five boxes, equals the quantity in eight boxes. The multiplications in $3x, 5x$, and $8x$ do represent these quantities, and this is a correct explanation of the meaning of the distributive property.

Irrationals

We have now sketched the construction of the rational number system: fractions and their arithmetic. (Strictly speaking we have only considered positive fractions, but it is easy enough to include the negative ones following the usual rules $x + (-y) = x - y$, $x(-y) = -(xy)$, and so forth.) This construction is, and should be seen as, a crowning achievement of middle-school mathematics. For the first time, students have a number system which is closed under all the basic arithmetic operations, in which all computations yield exact results (no approximations), and which might (depending on how the existence of irrational numbers turns out) be adequate for all the measurement needs of the sciences. It is unfortunate (from this viewpoint) but true that irrational numbers do exist: many segments cannot be measured by the process we have developed. A simple example is $\sqrt{2}$: the diagonal of a square does not have any fractional length when the side of that square is used as the unit. It is important that students truly understand why this is so. Rather than the usual formal proof based on prime factorization, here is a more conceptual one using geometry and measurement.

Suppose that the diagonal of a square did have a rational measure a/b in units of the side. Following the standard logic of an indirect proof, we will deduce a contradiction from this assumption, thus showing that it cannot be true. Measuring with a smaller unit $1/b$ instead, we obtain a square $ABCD$ having integer side length b as well as integer diagonal length $a > b$ (see Figure 2). In fact, we can draw infinitely many such squares, corresponding to all the equivalent forms of the fraction a/b. Of all these possibilities we will choose the one in lowest terms: we pick the smallest possible values of a and b. Then there is no square smaller than $ABCD$ whose side and diagonal are both integers. (Here we are using the fact that all squares are similar: they all have the same ratio a/b of diagonal to side.) As in the figure, draw a circle centered at A, of radius $AB = b$, cutting the diagonal AC at E. Since $AC = a$ and $AE = b$, the remainder $EC = a - b$. From E, draw a line perpendicular to the diagonal AC, meeting side BC at F. Then CEF is a right triangle containing an acute angle of 45°, so it is isosceles and $EF = EC = a - b$. The segments FE and FB are tangents to the circle from the external point F, so they are equal: $BF = EF = a - b$. Then the remaining segment of the side BC is $CF = b - (a - b) = 2b - a$. Since a and b are integers, so are the sides $a - b$ and $2b - a$ of the triangle CEF. But this triangle is half of a square $CEFG$ with diagonal CF, and this square is clearly smaller than the original one: its diagonal CF is smaller than the side BC of the original. This contradicts the fact that we began with the smallest possible square having integer side and diagonal. This contradiction, following from the assumption that the diagonal of a square is a/b of its side, shows that this assumption is false regardless of how a and b are chosen.

Variations on this argument show that $\sqrt{n}$ is irrational whenever n is not the square of another natural number. Irrational numbers are ubiquitous, actually outnumbering rational numbers in an appropriate sense.

Real Numbers

The focus of this paper is on rational numbers, but we can take a step in the direction of the real number system. Although there are lengths such as $\sqrt{2}$ which are not rational numbers, they can be approximated by rational numbers. Given any denominator b, for example 341, we can find the fractions having this denominator and closest to $\sqrt{2}$. In terms of measurement, we can make a ruler marked in 341ths of a unit and see where the diagonal of a unit square falls on it. Arithmetically, we can square fractions having various numerators until we find that

$$\frac{482}{341} < \sqrt{2} < \frac{483}{341}.$$

(That is, we find by trial and error that $482^2 < 2 \times 341^2$ but $483^2 > 2 \times 341^2$.) This means that 341 copies of the diagonal will be longer than 482, but shorter than 483, copies of the side. We can thus locate $\sqrt{2}$ in an interval of length 1/341 between two fractions, and we can similarly do this for any interval $1/b$. (It is important that students realize that they have the ability to approximate numbers like $\sqrt{2}$ themselves, with their existing mathematical knowledge, without appealing to tables or the teacher's authority. They may need a calculator, but only to perform well-understood operations like multiplication, not to magically extract square roots.) Now we ask a second philosophical question: if we do not know the exact measure of some length, but we can on demand find the nearest fractions with any desired denominator, what are we missing? What could we do with the exact value that we cannot do with the fraction approximations? The real number system was born when mathematicians answered, "Nothing." There is no theoretical or practical difference between knowing the exact value of a real number and knowing arbitrarily good approximations. Indeed, in many ways it is more realistic to work with approximations, acknowledging that no measurement is exact. Measurement with rational numbers requires exact judgments of whether segments are equal, which is unrealistic (in the physical, as opposed to the mathematical, world).

What about decimals? Fundamentally, decimals are simply an impoverished system of fractions in which we agree to only choose denominators b which are powers of ten. Thus, in place of 1/4 we choose the equivalent fraction 25/100, which we abbreviate as 0.25. Although this amounts to performing measurements with one hand tied behind our backs, it has some advantages due to standardization. If you and I choose different denominators for our fractions, it may be hard for us to compare results. It is harder to determine that $196/273 > 295/414$ than to make the same judgment about their decimal approximations $0.7179 > 0.7126$. Also, the standard arithmetic algorithms taught in elementary school are based on the place value system of decimal numerals. If the same place value system is used to represent fractions, then the same algorithms apply (apart from deciding where the decimal point goes). A price has to be paid: many fractions, for example 1/3, cannot be represented as finite decimals because there is no equivalent fraction whose denominator is a power of ten. To measure 1/3 of a unit, we could divide the unit into tenths and take three of them, obtaining the approximation 0.3. But this is too small: three of these approximate thirds only fills 9/10 of the unit, leaving 1/10. Dividing the unit into hundredths gives a closer approximate third of $33/100 = 0.33$ with 1/100 remaining, and so on. This is what is meant by $0.333\ldots = 1/3$: taking more decimal digits gives approximations to 1/3 which eventually become as accurate as desired. (Compare the similar statement $0.999\ldots = 1$, which students often disbelieve.) Pedagogically, the idea of approximating fractions such as 1/3 more and more closely by decimals can motivate the idea of approximating real numbers like $\sqrt{2}$ as well. A real number is completely known when all its finite decimal approximations are known. The statement $\sqrt{2} = 1.4142135\ldots$ can be interpreted this way: the right side represents the set of finite decimal approximations $\{1, 1.4, 1.41, 1.414, 1.4142, \ldots\}$. This set determines the unique real number $\sqrt{2}$.

References

The following references describe approaches to the teaching of fractions which have some similarity to the one I have presented. The article by Anne Morris is particularly interesting because it describes her experience using Davydov's measurement-based curriculum in a class of American fourth-graders. Lamon presents a wide variety of approaches, including measurement, while Wu's central theme is the positioning of fractions on the number line.

V. Davydov and Z. Tsvetkovich, On the objective origin of the concept of fractions, *Focus on Learning Problems in Mathematics,* 13(1) (1991) 13-64.

A. Morris, A teaching experiment: Introducing fourth graders to fractions from the viewpoint of measuring quantities using Davydov's mathematics curriculum, *Focus on Learning Problems in Mathematics,* 22 (2000) 32-83.

S. Lamon, *Teaching Fractions and Ratios for Understanding,* Erlbaum, New Jersey 1999.

H. Wu, Chapter 2: Fractions, at: math.berkeley.edu/˜wu/.

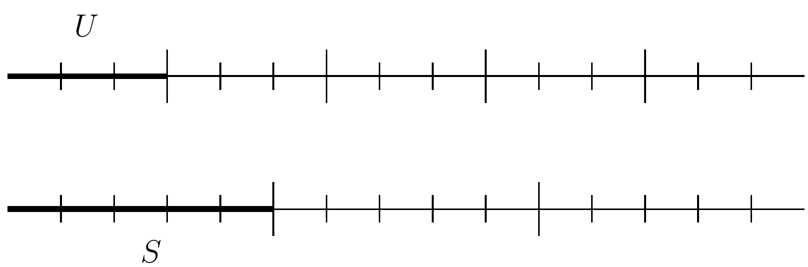

Figure 1. Segment S equals 5/3 of the unit U.
Three copies of S equals five copies of U.
If U is divided into three equal parts, S equals five of them.

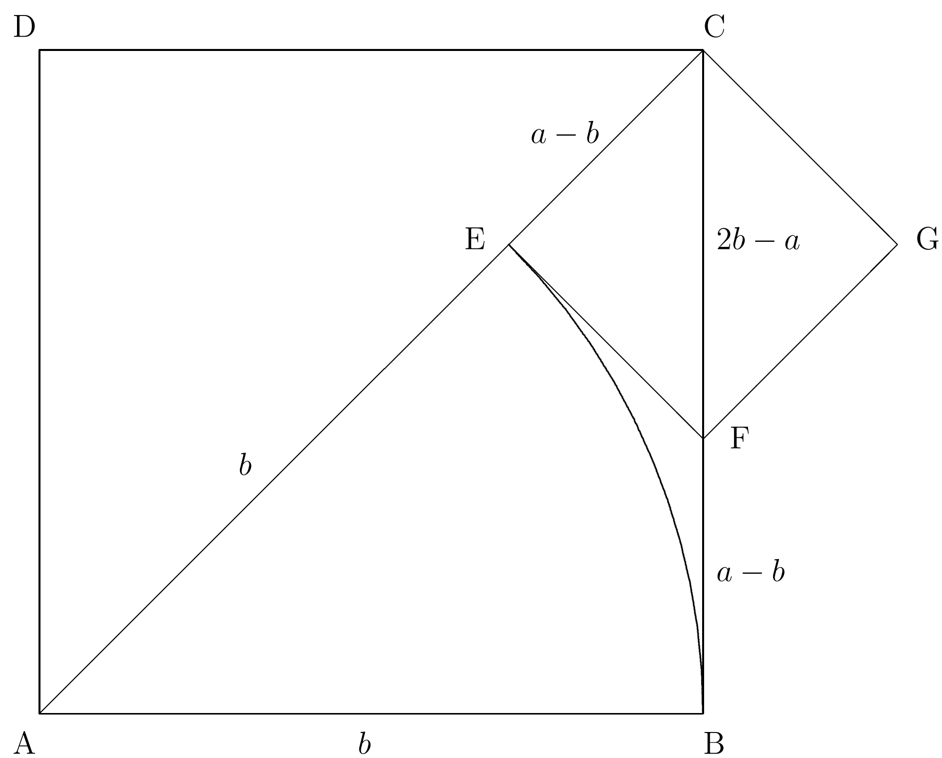

Figure 2. Proof that $\sqrt{2}$ cannot be rational.