# ON DECOMPOSING $N = 2$ LINE BUNDLES AS TENSOR PRODUCTS OF $N = 1$ LINE BUNDLES

Fausto Ongay[1] and Jeffrey M. Rabin[2]

[1]CIMAT, Guanajuato, Gto. 36240 Mexico
[2]Dept. of Mathematics, UCSD, La Jolla CA 92093 USA
e-mail: ongay@cimat.mx; jrabin@ucsd.edu

Abstract. We obtain the existence of a cohomological obstruction to expressing $N = 2$ line bundles as tensor products of $N = 1$ bundles. The motivation behind this paper is an attempt at understanding the $N = 2$ super KP equation via Baker functions, which are special sections of line bundles on supercurves.

There has been—for some time now (cf. [DG])—an interest in extending the study of the super KP equations from the case $N = 1$ to the case $N = 2$. One possible way to do this, that could be particularly useful for understanding the geometry of these equations, would be via Baker functions: Roughly speaking, Baker functions are special unique sections with parameters of certain families of line bundles, satisfying the condition that any section of the corresponding line bundle is given by a differential operator applied to the Baker function. From the geometric point of view, their relevance stems from the fact that they allow us to reinterpret equations such as the KP (or its super analogs), as describing deformations of line bundles over curves (or supercurves).

Thus, as a necessary step towards such a study of the super KP equations, one must first understand the geometry of $N = 2$ superline bundles, and this note is aimed towards that goal. Specifically, what we obtain here is the existence of a cohomological obstruction to expressing $N = 2$ line bundles as tensor products of

1991 *Mathematics Subject Classification.* 14M30.
*Key words and phrases.* Complex supercurves, complex superline bundles, super Riemann surfaces.
Partially supported by CONACYT, Mexico, project 37558-E.

$N = 1$ bundles. (And therefore, in general there might not be a simple relationship between $N = 1$ and $N = 2$ Baker functions.)

To properly describe our result, let us first recall that a smooth (complex) supercurve $M$ is a family of ringed spaces $(M_{\text{red}}, \mathcal{O}_M)$, over the base $(\bullet, \Lambda)$. Here $\Lambda$ is a Grassmann algebra (in some generators, say $\eta_1, \ldots, \eta_n$), $M_{\text{red}}$ is an ordinary smooth complex curve, and the structure sheaf $\mathcal{O}_M$ is a sheaf of $\mathbb{Z}_2$-graded algebras, locally isomorphic (as sheaves of $\mathbb{Z}_2$-graded algebras) to $\mathcal{O}_{\text{red}} \otimes \Lambda[\theta^1, \ldots, \theta^N]$, $\mathcal{O}_{\text{red}}$ being the structure sheaf of $M_{\text{red}}$, and $\theta^\alpha$ odd generators nilpotent of order two. (Super)line bundles over the supercurve are then defined as locally free sheaves of rank 1 $\mathcal{O}_M$-modules.

Supercurves can be described in a more concrete way, by prescribing the changes of coordinates between charts: for the case $N = 1$, where we simply write $\mathcal{O} = \mathcal{O}_M$, this takes on the form

$$\begin{aligned} z_j &= f_{ji}(z_i) + \theta_i \gamma_{ji}(z_i) \\ \theta_j &= \mu_{ji}(z_i) + \theta_i n_{ji}(z_i), \end{aligned}$$

where $f_{ji}$, $n_{ji}$ are even invertible holomorphic functions, while $\gamma_{ji}$, $\mu_{ji}$ are odd. Line bundles over $M$ (i.e., $N = 1$ bundles) are then determined by transition functions $\Gamma_{ji} = a_{ji} + \theta_i \alpha_{ji}$, with $\Gamma_{ji}$ an invertible superfunction satisfying a cocycle condition.

To any such supercurve $M$ there is an associated $N = 2$ super Riemann surface (SRS), denoted $M_2$, constructed by adjoining an odd coordinate $\rho$ that transforms according to

$$\rho_j = \frac{\gamma_{ji}}{n_{ji}} - \theta_i \rho_i \left( \frac{\gamma_{ji}}{n_{ji}} \right)' + \rho_i \frac{f'_{ji} n_{ji} - \mu'_{ji} \gamma_{ji}}{n_{ji}^2},$$

the coefficient of $\rho$ being the Berezinian of the change of coordinates on $M$; we denote the structure sheaf of $M_2$ by $\mathcal{O}_2$. Moreover, by considering $\rho_i$ as the odd coordinate and $\hat{z}_i = z_i - \theta_i \rho_i$ as the even coordinate one gets, associated to $M$, another $N = 1$ supercurve, $\hat{M}$, called the *dual* supercurve, whose structure sheaf we denote as $\hat{\mathcal{O}}$.

Recall that any $N = 2$ super Riemann surface has a global rank $0|2$ nonintegrable distribution, locally generated by $D^+ = \partial_\rho$ and $D^- = \partial_\theta + \rho \partial_z$. However, $M_2$ as constructed above is always an "untwisted" $N = 2$ SRS, meaning that each of $D^\pm$ generates a rank $0|1$ distribution. The sections annihilated by $D^+$ are called *chiral* sections, those annihilated by $D^-$ *antichiral*. Moreover, the structure sheaf of $M_2$ fits into the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}_2 \to \mathcal{B}er \to 0, \tag{1}$$

where $\mathcal{B}er$ denotes the Berezinian sheaf of $M$, and there is a similar sequence for $\hat{M}$, namely

$$0 \to \hat{\mathcal{O}} \to \mathcal{O}_2 \to \hat{\mathcal{B}}er \to 0.$$

The corresponding projections in these sequences are just the operators $D^+$ and $D^-$, respectively.

Observe that a line bundle over either $M$ or $\hat{M}$ can be viewed as a line bundle over $M_2$ having the same transition functions (these are the pullbacks via the obvious projections $M_2 \to M$ and $M_2 \to \hat{M}$), and their tensor product is a line bundle over $M_2$ as well. Conversely, given a line bundle on $M_2$ we can ask when it splits as a tensor product of line bundles on $M$ and $\hat{M}$.

To answer this question, first of all we make the observation that any $N = 2$ local superfunction,

$$\Gamma = h + \theta\phi + \rho\psi + \theta\rho g,$$

can be decomposed as a product of a function on $M$ and a function on $\hat{M}$, i.e.,

$$\Gamma = \big(a(z) + \theta\alpha(z)\big)\big(b(\hat{z}) + \rho\beta(\hat{z})\big).$$

In fact,

$$h = ab\ ;\ \phi = \alpha b\ ;\ \psi = a\beta\ ;\ g = -(ab' + \alpha\beta),$$

where now all functions depend on $z$. Since we can obtain $b'/b$ from them, these conditions actually determine $b$ up to a multiplicative constant (and $\beta$ also); $a$ is then determined up to the *reciprocal* constant (and so is $\alpha$).

**Remark**. There is a similar decomposition of $N = 2$ local superfunctions as sums of $N = 1$ superfunctions, $\Gamma(z, \theta, \rho) = F(z, \theta) + \hat{F}(\hat{z}, \rho)$, modulo an additive constant. This, of course, is essentially "taking the logarithm" of the decomposition above, and is a manifestation of the existence of an exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0,$$

where, as usual, $^\times$ denotes the invertible elements; however, we will not need this construction in what follows.

Now, if $\Gamma_{ji}$ are the transition functions of an $N = 2$ line bundle, written as above, since the $a_{ji}$ and $b_{ji}$ themselves are only determined up to a constant, say $c_{ji}$, what this actually gives is the existence of a short exact sequence of the form

$$0 \to \Lambda^\times_{\mathrm{ev}} \to \mathcal{O}^\times_{\mathrm{ev}} \times \hat{\mathcal{O}}^\times_{\mathrm{ev}} \to \mathcal{O}^\times_{2,\mathrm{ev}} \to 0, \tag{2}$$

where $\mathcal{O}_{\text{ev}}^{\times} \times \hat{\mathcal{O}}_{\text{ev}}^{\times}$ is the sheaf described at the presheaf level by taking direct products of the groups $\mathcal{O}_{\text{ev}}^{\times}(U_i)$ and $\hat{\mathcal{O}}_{\text{ev}}^{\times}(U_i)$. The second arrow is the map $c \mapsto (c, c^{-1})$ and the third is $(F, \hat{F}) \mapsto F\hat{F}$. Notice that all objects appearing in the construction of the sequence above are *even*; in particular the sheaves are sheaves of abelian groups.

We can now state our main result:

**Theorem.** *Let $M$ be an $N = 1$ supercurve. Then for any given line bundle $\mathcal{L}$ over the associated $N = 2$ SRS $M_2$, there exists a cohomology class in $H^2(M_{\text{red}}, \Lambda_{\text{ev}}^{\times})$, that measures the obstruction to expressing $\mathcal{L}$ as a tensor product of line bundles over $M$ and $\hat{M}$.*

*Proof*: Recall that, in general, line bundles over a complex (super) manifold are classified by the first cohomology group, $H^1(M_{\text{red}}, \mathcal{O}_{\text{ev}}^{\times})$.

Now, taking the cohomology sequence of the short exact sequence (2) above, the last part reads

$$\cdots \to H^1(M_{\text{red}}, \mathcal{O}_{\text{ev}}^{\times} \times \hat{\mathcal{O}}_{\text{ev}}^{\times}) \to H^1(M_{\text{red}}, \mathcal{O}_{2,\text{ev}}^{\times}) \xrightarrow{\beta} H^2(M_{\text{red}}, \Lambda^{\times}) \to 0. \tag{3}$$

However, there is a canonical map of cohomology groups

$$H^1(M_{\text{red}}, \mathcal{O}_{\text{ev}}^{\times}) \times H^1(M_{\text{red}}, \hat{\mathcal{O}}_{\text{ev}}^{\times}) \to H^1(M_{\text{red}}, \mathcal{O}_{\text{ev}}^{\times} \times \hat{\mathcal{O}}_{\text{ev}}^{\times}),$$

and this map is in fact an isomorphism, because the domain and range are both defined by equivalence relations on pairs of cocycles of transition functions. Indeed, the isomorphism is (essentially) determined by the assignment $\big(F(z), \hat{F}(\hat{z})\big) \mapsto \big(F(z), \hat{F}(z) - \theta\rho\hat{F}'(z)\big)$.

Therefore, we have a group morphism

$$\delta\colon H^1(M_{\text{red}}, \mathcal{O}_{\text{ev}}^{\times}) \times H^1(M_{\text{red}}, \hat{\mathcal{O}}_{\text{ev}}^{\times}) \to H^1(M_{\text{red}}, \mathcal{O}_{2,\text{ev}}^{\times}),$$

whose image, by construction, consists of the isomorphism classes of $N = 2$ bundles that can be expressed as tensor product of bundles over $M$ and $\hat{M}$. Thus, a line bundle over $M_2$ can be decomposed as a tensor product if and only if the corresponding class is in the image of $\delta$, which equals the kernel of $\beta$ in (3); this gives the cohomology obstruction whose existence was asserted. □

Let us make some remarks in relation to this result.

First of all, a somewhat more concrete proof of the theorem can be given along the following lines: Given a cocycle of transition functions for an $N = 2$ line bundle $\mathcal{L}$, recalling the sequence (1), we can form the quotients

$$\frac{D^+\Gamma_{ji}}{\Gamma_{ji}}\ ; \quad \frac{D^-\Gamma_{ji}}{\Gamma_{ji}},$$

and these are in fact sections of the Berezinian sheaves $\mathcal{B}er(M)$ and $\hat{\mathcal{B}}er(M)$ of $M$ and $\hat{M}$ respectively. By indefinite integration of these local sections and then exponentiation, up to multiplicative constants one gets functions $F_{ji}$ and $\hat{F}_{ji}$, and this gives a factorization of the form $\Gamma = F\hat{F}$ for the transition functions of $\mathcal{L}$ (see [BR] for details). Now, the cocycle conditions for $\Gamma_{ji}$ show that on triple overlaps one has

$$\log F_{ji} + \log F_{kj} + \log F_{ki} = \log \hat{F}_{ji} + \log \hat{F}_{kj} + \log \hat{F}_{ki},$$

modulo some integers (representing the Chern class of the bundle $\mathcal{L}$). But the left hand side is chiral while the right hand side is antichiral, so that they are necessarily equal to a constant, say $c_{kji}$; then $\exp c_{kji}$ gives an explicit representative for the desired class in $H^2(M_{\rm red}, \Lambda_{\rm ev}^\times)$.

On the other hand, the proof given above sheds some light on an interesting phenomenon. Namely, it was known that for certain supercurves (nonprojected generic SKP curves, cf. below) there are examples of nontrivial $N = 1$ bundles $\mathcal{L}$ giving a nontrivial factorization $\mathcal{O}_2 = \mathcal{L} \otimes \hat{\mathcal{O}}$, in addition to the trivial one, $\mathcal{O}_2 = \mathcal{O} \otimes \hat{\mathcal{O}}$; that is, $\mathcal{L}$ is nontrivial as a bundle over $M$, but its lift to $M_2$ is trivial [BR].

This can be explained as follows:

By going one further step backwards in the cohomology sequence (3), we get an exact sequence of the form

$$\cdots \to H^1(M_{\rm red}, \Lambda_{\rm ev}^\times) \to H^1(M_{\rm red}, \mathcal{O}_{\rm ev}^\times) \times H^1(M_{\rm red}, \hat{\mathcal{O}}_{\rm ev}^\times) \to H^1(M_{\rm red}, \mathcal{O}_{2,{\rm ev}}^\times) \to \dots$$

By construction, the second arrow in this sequence maps a flat line bundle (or, to be precise, its isomorphism class), say $\mathcal{L}$, defined by the cocycle of constant transition functions $c_{ji}$, to the pair of bundles $(\mathcal{L}, \mathcal{L}^{-1})$, defined by the pair $(c_{ji}, c_{ji}^{-1})$, where the bundles of the pair are regarded as being over $M$ and $\hat{M}$ respectively.

Thus, the assertion above is that the cocycle $c_{ji}$ might be trivial when seen as representing an element of $H^1(M_{\rm red}, \mathcal{O}_{\rm ev}^\times)$ (i.e., as defining a bundle over $M$), *but not* when seen as an element of $H^1(M_{\rm red}, \hat{\mathcal{O}}_{\rm ev}^\times)$ (i.e., as defining a bundle over $\hat{M}$), or vice versa. But this might very well happen, because in general $H^1(M_{\rm red}, \mathcal{O}_{\rm ev}^\times)$ and $H^1(M_{\rm red}, \hat{\mathcal{O}}_{\rm ev}^\times)$ are both quotients of a free rank $g$ $\Lambda$-module (where $g$ is the genus of $M_{\rm red}$, see [BR] for the calculation of these dimensions), and the two quotients *are different in general*. Moreover, this also shows that the bundles allowed in a nontrivial decomposition of $\mathcal{O}_2$ are necessarily flat (hence of degree zero).

Finally, let us point out that one can gain some further insight into the meaning of this cohomological obstruction by considering the relatively simple but important case of generic SKP curves (which are the curves needed for studying the super KP equation); these are curves for which the structure sheaf of the associated split curve has the form $\mathcal{O}_{\text{red}}|\mathcal{N}$ (recall that a vertical bar means "direct sum of even and odd parts"), with $\deg \mathcal{N} = 0$, but $\mathcal{N} \neq \mathcal{O}_{\text{red}}$. A simplifying feature of these curves is that they have free cohomology, so in this case $h^1(M, \mathcal{O}) = g|g-1$, while, as already mentioned, $H^1(M_2, \mathcal{O}_2)$ and $H^1(\hat{M}, \hat{\mathcal{O}})$ are in general only quotients of free $\Lambda$-modules, of ranks $g+1|g-1$ and $g|0$ respectively.

The space of line bundles of degree zero on $M$ can be identified with the quotient $H^1(M, \mathcal{O}_{\text{ev}})/H^1(M_{\text{red}}, \mathbb{Z})$, and similarly for $\hat{M}$ and $M_2$. Note that elements of $H^1(M, \mathcal{O}_{\text{ev}})$ are linear combinations of both the $g$ even generators of $H^1(M, \mathcal{O})$ with coefficients from $\Lambda_{\text{ev}}$, and the $g-1$ odd generators with coefficients from $\Lambda_{\text{odd}}$. On $M_{\text{red}}$ the space of degree-zero bundles is $g$-dimensional, and they can all be given by constant ($\mathbb{C}^\times$-valued) transition functions. Correspondingly, the bundles having constant ($\Lambda_{\text{ev}}^\times$-valued) transition functions account for the even generators of $H^1(M, \mathcal{O})$ and all of $H^1(\hat{M}, \hat{\mathcal{O}})$. Lifting bundles of degree zero from $M$ to $M_2$ accounts for at most those bundles coming from (a quotient of) a rank $g|g-1$ $\Lambda$-module in $H^1(M_2, \mathcal{O}_2)$, and lifting the degree-zero bundles from $\hat{M}$ adds nothing new. Therefore there is an "extra" bundle on $M_2$ which does not come from lifting bundles of degree zero on $M$ or $\hat{M}$, nor does it factor into these. It can be viewed as the generator of the group $H^1(M, \mathcal{B}er)$ in the long cohomology sequence associated to (1). This group has rank $0|1$, being Serre dual to $H^0(M, \mathcal{O})$ in the category of supercurves.

To identify this bundle, consider a covering of $M_{\text{red}}$ consisting of an open disk $D$ centered at a point $P$, with coordinate $z = 0$, and $M_{\text{red}} \setminus \{P\}$; then the extra line bundle has as transition function in the annulus

$$1 - k\frac{\theta\rho}{z} = \frac{1}{z^k}(z - \theta\rho)^k. \tag{4}$$

That this has to be the form of the transition function can be justified by the fact that the residue mapping is integration of the principal parts representing elements of $H^1(M, \mathcal{B}er)$, so the dual of a constant function $\kappa$ should represent a bundle having a transition function with a pole of the form $\kappa/z$ (this is an odd morphism, which explains why we get even bundles; again see [BR] for the details on these constructions).

The point is that when $k$ is not an integer, $z^k$ *is not* single valued in $D \setminus \{P\}$ so, in these cases, the functions appearing in the decomposition given by the right hand side of (4) cannot define line bundles over $M$ and $\hat{M}$, and one sees that some (indeed, most) of these bundles cannot come from tensor products of bundles. Also, observe that the $N = 2$ bundles appearing in this situation all have degree 0, but in case $k$ is an integer, this gives a decomposition into $N = 1$ bundles of degrees $k$ and $-k$.

**Acknowledgement.** F. Ongay wishes to warmly thank the Department of Mathematics at UCSD, where the first version of this work was written, for its kind hospitality.

## References

[BR] Bergvelt, M. J. and J. M. Rabin, *Supercurves, their Jacobians, and super KP equations*, Duke Math. Journal **98, No. 1** (1999), 1-57.

[DG] Delduc, F. and L. Gallot, $N = 2$ *KP and KdV hierarchies in extended superspace*, Commun. Math. Phys. **190** (1997), 395-410.