Double Negative: The Necessity Principle, Commognitive Conflict, and Negative Number Operations

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Abstract: We use the DNR framework to analyze a classroom episode introducing negative integer exponents, comparing and contrasting our analysis with Sfard's recent commognitive analysis of a similar episode concerning multiplication of signed numbers. Students in both episodes objected to the standard rules for integer products or exponents, and they persisted in preferring their own rules even after the teacher justified the standard ones. We examine how pattern-based justifications may not address students' intellectual needs, and we suggest other pedagogical strategies that promote student reasoning.

Keywords: conventions, didactical obstacles, DNR system, Necessity Principle, negative number operations, negative exponents.

1. Introduction.

Students face many challenges as they confront new mathematical ideas, especially ideas that extend the scope of previously secure knowledge, or require its modification. Brousseau

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(1997) distinguished between *epistemological obstacles* and *didactical obstacles* in this regard. Epistemological obstacles are inherent in resolving tensions between the new material and existing conceptual schemes, and may exhibit parallels with the historical path of development of the new ideas. Didactical obstacles result from the particular teaching strategies employed, either in presenting the new material or earlier in laying the foundations of the existing knowledge. Particular didactical obstacles might thus be avoided by alternative pedagogical choices. Epistemological obstacles cannot be so avoided, but can actually be beneficial for developing students' mathematical thinking.

New material that students learn might take the form of *theorems* deducible from current knowledge, or might consist of definitions or conventions logically independent of it[4]. For example, the Pythagorean Theorem follows from appropriate geometric foundations, but the definition of a zero exponent is a pure convention. One might define the term *convention* so broadly that it includes all mathematical choices that are not forced upon us logically, including the choices of foundational axioms and definitions. However, this is far more inclusive than our present interest. For us, a convention is an agreement about what a mathematical term or notation will mean, often extending the scope of an existing term like "multiplication" to a larger set of numerical arguments in a way that preserves certain properties. For example, even after the concept "factorial" is defined, the notation 0! is still meaningless. The convention that $0!=1$ simplifies certain formulas. Another example of a convention is that $\sqrt{x}$ denotes the *positive* square root. Teachers may create didactical obstacles, such as confusion between definitions and

[4] The term *theorem* may seem pretentious at the middle school level, but we adopt it to stress the distinction between what is provable and what is agreed upon by convention.

theorems, if they are not aware of which category the new material belongs to and/or do not make their students aware of it.

The distinction between theorems and conventions is actually more subtle than it initially appears. For example, a theorem may depend on previously adopted conventions: the theorem that every integer greater than 1 has a unique prime factorization depends on the convention that 1 is not considered to be a prime number. For a useful analogy, consider a game such as chess. The rules of chess are conventions, adopted because they lead to an interesting and enjoyable game. They did not have to be what they are, and indeed they evolved over time. However, once these rules are adopted, there are objective facts (theorems) about chess. For example, a king and queen can force checkmate against a lone king. Mathematics develops, both historically and in the classroom, in a cycle of adopting conventions, proving theorems about their consequences, adopting further conventions, and so on. It is a legitimate part of this process to "look forward", exploring the consequences of alternative possible conventions before deciding which one to adopt. Furthermore, what is a convention in one instructional treatment of some topic may be a theorem in another: when there are equivalent definitions of the same concept, either one can be adopted as "the" definition and the other is then a theorem. Within a particular instructional treatment, however, it should be clear what is agreed upon as a convention and what is justified as a theorem. Students (and teachers) are often unaware of the major role of conventions in mathematics, and we believe this role should be made more explicit in the classroom.

In this paper we are concerned with the introduction of new material in one topic area: negative integers and arithmetic operations on them. This involves a dramatic extension of the previously developed whole number system. It depends on specific conventions, and it presents both epistemological and didactical obstacles. Our discussion focuses on a classroom episode in which the teacher introduces negative integer exponents. We compare and contrast it with similar episodes involving multiplication of signed numbers as presented and analyzed by Sfard (2007).

Sfard's analysis is based on her theoretical framework of commognitive conflict. Learning about negative numbers is framed as the acquisition of a new discourse that is *incommensurable* with the old discourse of natural numbers. The terminology recalls Kuhn's analysis of scientific revolutions (Kuhn, 1970), according to which these result in such a dramatic revision of concepts and terminology from the old science to the new that statements in one scientific language cannot even be translated into the other, let alone arbitrated by some common set of criteria. Therefore Sfard seems to view the epistemological obstacles involved in this negative number "revolution" as far outweighing any didactical obstacles: "As implied by the commognitive analyses, the difficulties revealed on these pages, rather than being an unintended result of particular instructional approaches, were part and parcel of the process of learning" (Sfard 2007, p. 612). We think this implicit labeling of the difficulties revealed by her analysis as epistemological leads to an undue pessimism about the potential for didactical improvement.

Our own analysis is based on the DNR theoretical framework (Harel, 2008a, 2008b), as outlined below (the initials stand for the three central principles of the framework: *D*uality, *N*ecessity, and *R*epeated Reasoning). Although there are common elements in the two analyses, we place more weight on the potential for controlling didactical obstacles. Extending arithmetic to include negative integers does require adopting conventional definitions, but these can be motivated (*necessitated*, in DNR terminology) on the basis of existing knowledge and problematic situations (and therefore need not be accepted purely on authority). Thus, we see more continuity than incommensurability between the existing knowledge and the new material. We use the Necessity Principle to make conjectures about students' intellectual needs relevant to learning about negative numbers, and to suggest alternative instructional treatments addressing these needs.

Our work makes several contributions. First, we extend the limited research on teachers' understanding of the distinction between mathematical conventions and theorems (Levenson, 2012) and corresponding pedagogical strategies. Second, despite copious anecdotal evidence that teachers and students alike are mystified by the rules for signed number operations, there is surprisingly little research on the basis for their difficulties (one example is Thompson & Dreyfus, 1988). We apply the Necessity Principle of the DNR system to conjecture the intellectual needs underlying some student difficulties. Indeed, both our episode and Sfard's feature students objecting on intellectual grounds to the justifications presented by their teachers, which we interpret as evidence that these justifications do not address their intellectual needs. Third, teachers often make use of numerical patterns to justify mathematical claims. We confirm Sfard's observation that such justifications may not be convincing to students, or even understood by them as justifications. Our analysis indicates why pattern-based justification may not address students' needs and what strategies might improve on it. Finally we provide examples of how an analysis based on the Necessity Principle can lead to concrete pedagogical recommendations.

2. The Signed Number Multiplication Episodes

In examining mathematical discourse, Sfard (2007) analyzed teaching episodes concerning the rule for multiplication of signed numbers. These took place in an Israeli junior secondary school in a class of 12- to 13-year-old students. The class was observed over thirty one-hour meetings devoted to the topic of negative numbers, and the discussion of multiplication of signed numbers spanned several class periods. The observations of interest began when students were given the task of deciding the value of the product of a positive number with a negative number, e.g. $(+2) \times (-5)$. One group decided that multiplication by +2 means adding the other number to itself, so the answer is -10. However, another student, Roi, argued that the unsigned product 10 should always be given the sign of the "bigger number" (in absolute value). This leads to the same answer in this case, but for example $(+7) \times (-5)$ would be +35 (since 7>5)[5]. A class discussion ensued, and not only did the students fail to collectively agree on the "correct" rule as the teacher had expected, but the majority endorsed Roi's proposal. We note that the teacher had provided students with three visual representations to support their thinking: the number line, arrows (vectors on the number line), and "magic cubes" which increase or decrease the temperature of a liquid they are added to by one degree. However, these representations did not seem to play any significant role in the students' thinking or argumentation in the episodes quoted. Ultimately, the teacher resolved the debate using her own authority:

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[5] Sfard suggests that Roi is generalizing from a definition of addition of a positive and a negative number: subtract the unsigned numbers, and attach the sign of the bigger number.

T: I want to explain what Sophie [an advocate of the "correct" rule] said. What she said is true … and this is the right answer.

During a subsequent class, students were asked to determine the value of a product of two negative numbers, e.g. $(-3) \times (-2)$. Without a positive factor, there is no interpretation in terms of repeated addition, and most students could not obtain an answer, although Roi's rule is still applicable (and incorrect). The teacher then presented a "derivation" based on generalizing a pattern:

$2 \times 3 = 6$

$2 \times 2 = 4$

$2 \times 1 = 2$

$2 \times 0 = 0$

Continuing this pattern, one should conclude that

$2 \times (-1) = -2$

$2 \times (-2) = -4$

and so forth. Having substantiated Sophie's rule for positive times negative, the teacher began anew with a similar pattern:

$3 \times (-3) = -9$

$2 \times (-3) = -6$

$1 \times (-3) = -3$

$0 \times (-3) = 0$

and "therefore"

$(-1) \times (-3) = +3$

$(-2) \times (-3) = +6$

and so forth: negative times negative is positive.

The response of the students to this attempted justification is interesting: they rejected it. One said:

> Shai: I don't understand why we need all this mess. Is there no simpler rule?

Sophie herself was dismissive:

> Sophie: And if they ask you, for example, how much is $(-25) \times (-3)$, will you start from zero, do $0 \times (-3)$, and then keep going till you reach $(-25) \times (-3)$?

That is, rather than interpreting the pattern as an attempt at justification, she viewed it as a needlessly cumbersome computational algorithm [to find $(-25) \times (-3)$, one has to list all products from $(-1) \times (-3)$ to $(-25) \times (-3)$, increasing the result by 3 each time]. Indeed, the teacher used the word "compute" to describe what she was doing at one point (ibid, p. 591). In the end, students remained confused about how to operate on negative numbers.

3. Theoretical Framework

Sfard's (2007) analysis of the preceding episodes is based on her commognitive ("communication" + "cognition") approach to the study of learning. Basic tenets of this approach include the following. Thinking is the individualized form of communication, that is, communication with oneself, and it originates in interpersonal communication. Mathematics is a form of discourse, and thus "Learning mathematics may now be defined as individualizing mathematical discourse, that is, as the process of becoming able to have mathematical communication not only with others, but also with oneself" (ibid, p. 573). Two types of such learning may be distinguished, namely object-level learning, which is the expansion of the existing discourse, and meta-level learning, which involves changes in the meta-rules of the discourse. This dichotomy recalls Kuhn's distinction between normal science and revolutionary science (Kuhn, 1970). Normal science applies established methods to solve well-defined problems, and provides criteria for recognizing acceptable solutions. In revolutionary science, fundamental definitions, procedures, and theories may change, and the resulting new science has been held to be *incommensurable* with the old. When student and teacher employ different discourses, the student encounters a *commognitive conflict*, "a situation in which communication is hindered by the fact that different discursants are acting according to different meta-rules (and thus possibly using the same words in differing ways)" (ibid, p. 574), which may eventually result in the student adopting the teacher's new discourse. Sfard endorses the view that such discourses are incommensurable, so that in a sense the student cannot have good intellectual reasons to adopt the new discourse as long as she remains within the conceptual framework of the old. Commognitive conflicts are not factual disagreements that could be resolved by appeal to objective features of the world or by mathematical proof, but rather disagreements about the adoption of conventions governing discourse. Thus, students must gradually accept the new discourse based on the teacher being an "expert interlocutor" and can only later figure out the "inner logic" of the new discourse.

Sfard observes that the topic of negative numbers is particularly likely to precipitate a commognitive conflict, one of the first such genuine conflicts that learners have experienced in mathematics. In their experience with natural numbers, mathematical claims were ultimately grounded in the properties of an obvious physical model, for example by counting discrete objects. Negative number operations lack such an obvious model[6], and discourse about them is governed (implicitly, for students at this level) by the choice of certain axioms (notably the distributive property) that this extension of the concept of number is required to preserve. This change in the meta-rules of discourse and justification creates a commognitive conflict.

[6] Models for negative number operations are readily available in the modern world – elevators, temperatures, credits/debits, and so forth – and these are used in many textbooks, but such models are initially incompatible with those that students have previously relied upon for natural numbers. Reconciling the new and old models requires reflective abstraction, to use Piaget's terminology. This is an epistemological obstacle.

Our analysis of a similar episode involving the adoption of rules for operations on negative numbers is based on the DNR theoretical framework (Harel, 2008a, 2008b). The premises and principles of this system are wide-ranging, and we summarize only those most relevant to our analysis. The most important for us is

> The Necessity Principle: For students to learn what we intend to teach them, they must have a need for it, where "need" means intellectual need, not social or economic need.

What makes this principle effective in the analysis and design of teaching and learning situations is an explicit list of types of intellectual need that have historically led to the creation of new mathematics (both content and methods) (Harel, 2008b; Harel, 2013), and that can be pedagogically fostered in the classroom (Fuller, Harel, & Rabin, 2011); A synopsis of this list follows:

> The Need for Certainty: the need for proof; to remove doubts or determine whether a claim is true or false.

> The Need for Causality: the need to explain; to understand what makes a phenomenon occur, or what makes a claim true. Note that there are proofs, for example by contradiction, or by exhaustively verifying a large number of cases, which arguably do not explain.

The Need for Computation: the need to quantify, to calculate exact or approximate values, as well as to improve the efficiency of algorithms.

The Need for Communication: the need to persuade others, to adopt unambiguous definitions and notations, to agree on standard forms of expressions, arguments, or algorithms, [in Sfard's terms] to agree on the meta-rules of discourse.

The Need for Connection and Structure: the need to organize knowledge into a structure, to generalize or subsume, to determine unifying principles or axioms.

Learning in DNR is defined as "a continuum of disequilibrium-equilibrium phases manifested by (a) intellectual and psychological needs that instigate or result from these phases and (b) ways of thinking and ways of understanding[7] that are utilized and newly constructed during these phases" (Harel, 2008b; Harel & Koichu, 2010). Thus, learning is not principally about communication, but about the construction of new knowledge in response to intellectual needs. To *necessitate* a piece of mathematical knowledge in DNR is to embed the knowledge in a problematic learning situation that appeals to or stimulates one or more of the listed intellectual needs. DNR-based instruction presumes as a working hypothesis that this can be done: students *can* develop intellectual reasons to extend their existing knowledge. This notion includes and clarifies what is often called "motivation" for adopting conventions. We will present concrete pedagogical suggestions for bringing this about in Section 6 below.

[7] *Ways of thinking* and *ways of understanding* are technical terms in DNR, linked by the Duality Principle. Roughly, ways of thinking are mathematical habits of mind, while ways of understanding are concrete products of mathematical activity such as theorems, proofs, or algorithms.

DNR also contains a typology of proof schemes, ways of thinking that individuals may use to satisfy their own need for certainty, even if the mathematical community does not regard them as conclusive or correct (Harel & Sowder, 1998). These include accepting claims on another's authority (authoritative proof scheme), appealing to observation, measurement, or a limited number of examples (empirical proof schemes), and various forms of logical reasoning (deductive proof schemes). As more specific examples, we mention Result Pattern Generalization (RPG), an empirical proof scheme in which, for example, the universal validity of a numerical pattern is accepted based on the verification of a limited number of examples (results), and Process Pattern Generalization (PPG), a deductive proof scheme in which the validity is established by reasoning on the basis of the process that generates the pattern (Harel, 2008a).

4. The Negative Exponents Episode

The new episode we will analyze comes from a larger study that we have discussed elsewhere (Harel, Fuller, & Rabin, 2008; Harel & Rabin, 2010). Classroom observations were made of several teachers who had participated in a DNR-based summer professional development program. This episode occurred in a tenth-grade Algebra 1 classroom in the southwestern United States, so the students were somewhat older (about age 15) and more advanced than those in Sfard's study. The teacher's goal was to introduce negative integer (and zero) exponents and the rules for working with them. Our data consist of videotapes (which were transcribed) of the class, and notes from a debriefing conversation with the teacher a few days after the lesson[8]. We did not interview the students or collect their written work. However, the camera zoomed in on about half a dozen students' papers, allowing us to read their work. The episode unfolds over about 25 minutes, half the class period.

Students had previously worked problems involving exponential growth, particularly repeated doubling, and the teacher wanted to lead them to the definition of negative integer exponents. Referring to a problem in the textbook, he said:

> T: What I'd like you to do is work with a neighbor...and finish the rest of that table. Let's see if you can figure out the rest of the numbers that go into it. Real quick...figure out the table.

The table in question is the following:

[8] The conversation, conducted by one of us, was designed to help the teacher reflect on his goals for the lesson and to what extent the classroom activities achieved them. As part of it, the teacher viewed some excerpts from the videotaped lesson.

$y = 2^x$	$y = 5^x$	$y = 10^x$
$2^2 = 4$	$5^2 = 25$	$10^2 = 100$
$2^1 = 2$	$5^1 = 5$	$10^1 = 10$
$2^0 = \square$	$5^0 = \square$	$10^0 = \square$
$2^{-1} = \square$	$5^{-1} = \square$	$10^{-1} = \square$
$2^{-2} = \square$	$5^{-2} = \square$	$10^{-2} = \square$

Based on his instructions, his behavior throughout this episode, and the debriefing conversation, it was clear that the teacher expected this activity to be easy, unambiguous, and convincing for the students ("Real quick"), perhaps because of their experience recognizing patterns and applying Result Pattern Generalization. More importantly, he seemed to view this as an instance of determining unique correct answers ("figure out the rest of the numbers") rather than agreeing on a mathematical convention. Students worked for about ten minutes as the teacher circulated to help them. Some were confused, but many completed the table according to the nonstandard (but visually appealing) pattern that $a^{-x} = -a^x$. Various students took a^0 to be 0, 1, or *a*. No worksheet visible in our videotape contained fractional entries as required by the "correct" pattern, and no student advocated fractional values during the class discussion.

From the viewpoint of RPG, the students' pattern is as logical as the teacher's intended answer. The limited set of examples in the table allows many plausible generalizations. While working with individual students, and then while addressing the entire class, the teacher repeatedly directed their attention to a table of positive powers of 2 [extended from the first column of the textbook table; note the similarities with Sfard's last episode], pointing out the pattern that 2^x is halved when x decreases by 1 and explicitly directing them to continue this pattern. However, they still resisted extending the pattern to fractional entries, and the teacher's growing frustration became apparent. Eventually, he tried an independent visual source of justification, using a computer projector:

T: Okay, let me get your attention for a second. Maybe this will illustrate it in a different way. What I've done on the screen is I've graphed a function. It's $y = 2^x$.

The graph he displayed was the standard one that any calculator would produce for this function, with the entire real axis as domain: a continuous curve extending to negative as well as positive values of x, having positive slope and concave up. He pointed out that indeed the graph shows that decreasing x by 1 halves y, even when x is negative. This appeal to the empirical proof scheme (visual evidence) still did not convince students, who argued, for example:

S1: I'm confused because I don't understand how the 2 and the 1 [presumably meaning the negative exponent -1] equal ½. I thought that would be a -2. Because I'm confused.

T: Okay, that's logical reasoning.

S1: Even though that's a pattern it doesn't work.

T: What's happening each time? This is getting halved each time, right?

S1: I don’t understand that. I don’t understand how 2 to −1 could equal ½?

S2: Yeah, it looks like it was going to be a negative.

T: Can you see what was happening on the curve over here? Regardless of where I looked at the curve.

S1: I know, but when we did two positive ones you got 2, so when you get two negative ones would be negative [some students still use this language for exponents, so e.g. “two positive ones” seems to mean 2^{+1}].

S2: That’s what I thought, too.

T: You thought this [y coordinate for $x = -1$] would be a −2?

S2: Yes, because it’s −1.

T: [considering how the graph would look if the student were right] So it would go from, our line would come down here, and when it went to −1, all of a sudden it jumps down to −2, −4.

S1: [inaud]

T: Okay, let me see if I can think of a reason why it doesn’t do that. Let me go a little bit longer here and see if you can accept what I’m describing.

At the board, the teacher filled in the empty cells in the table, emphasizing the “correct” pattern of repeated division by 2, 5, or 10. Students saw the pattern but were still not convinced:

> S: I get the pattern and why you’re doing it, it’s just dividing itself. But I don’t understand, I don’t know. … But how does 2? Forget it. I just don’t get how it could go like that.

Another student seemed to be confusing 2^{-x} with $(-2)^x$, because she expected the sign to alternate for even and odd values of *x*. Of course, that would be another plausible candidate definition.

The disagreement continued, and eventually the teacher had to present the "correct" definition $a^{-x} = 1/a^x$ purely on his own authority, giving the social need:

T: This is what you need for your homework.

Various explanations can be suggested for the students' difficulties in accepting the teacher's desired answer. On the simplest level, they may be confusing 2^{-x} with $(-2)^x$ or even $2(-x)$. They may assume that a sign change in a problem causes a sign change in the answer, or they may be avoiding fractions because of discomfort with them. On the other hand, some of their statements suggest a Need for Causality, or for Computation. Despite seeing the pattern, S1 does not understand "how the 2 and the 1 equal ½": what formula or process with inputs 2 and 1 gives the output ½? (The fragment "But how does 2?" may express the same need.) That is, students believe there must be a rule for *computing* 2^x when *x* is negative, involving 2, *x*, and multiplication, as there is when *x* is positive. Since division is not involved, such a rule should give a negative integer result, not a fraction. Merely fitting into a pattern in a table does not sufficiently explain what *causes* 2^x to have a particular value; only a computational rule can do so for them. Although $a^{-x} = 1/a^x$ is a computational rule, it does not seem satisfying to students. We conjecture that this is because students' understanding of exponents involves multiplication rather than division, so any computational formula for exponents should involve multiplication but not division. We note that Sfard's students also seemed uncomfortable with the pattern-based justification (which, in that case, they interpreted as an unnecessarily cumbersome algorithm), preferring Roi's computational rule. Sfard (p. 593) cites Roi's comment that "there must be a law, one rule or another" as evidence that the students' discourse assumes that "whenever one dealt with entities called *numbers*, there had to be formulas that would tell one what to do". We interpret this as the same sort of Need for Computation that we identify in our episode.

We emphasize that our attribution of a Need for Causality or Computation to the students is conjectural. Methodologically, the best evidence for an intellectual need is what eventually satisfies that need, or could satisfy it (Harel, 2013). Since the students' intellectual needs were not satisfied during the lesson, and in the absence of interview data, we offer our own interpretation of the students' statements and behavior.

The teacher in our study did an admirable job involving students as arbiters of correctness, but he was hampered by his belief that what he had to justify is a *theorem* when it is actually a *convention* (if this distinction existed for him). This convention is adopted so that the law of exponents $a^{x+y} = a^x a^y$ will hold in greater generality, but this law played no explicit[9] role in the class discussion. Consequently, the teacher had no deductive argument for preferring his desired pattern to those advocated by students. The teacher appealed to the graph of $y = 2^x$, which he displayed for the class. During the debriefing conversation, he said, "I didn't think it was going to be that difficult. I had a graph on the sketch pad so I could show them that *y* is halving all the way down through. And to me, walking into the classroom, this is the evidence, how can they argue with that?" This justification is purely empirical and authoritative. So far, in

[9] This law is implicit in the teacher's insistence that 2^{x-1} should always be half of 2^x, although he may not be aware of the connection.

this classroom, the function 2^x is defined for natural numbers *x* only. What is at issue is precisely how to define it for negative (let alone rational or real) *x*. A full graph cannot be drawn until *after* this issue is resolved. When a student questions why the graph couldn't look different, the teacher takes this objection seriously and acknowledges that he has no convincing reason. However, the logical relationship between graphs and functions is never clarified: a function must be defined *before* it can be graphed (unless one takes the graph itself as the definition).

We note that reflection on the meaning of exponents and how this changes to admit negative exponents could be another means by which to persuade students to adopt the correct rule. One could alter the interpretation of exponents so that while a^n for a natural number *n* means $a \times a \times \ldots \times a$ (*n* times), a^{-n} would mean "undoing this:" i.e. the number that, when multiplied by $a \times a \times \ldots \times a$ (*n* times) gives 1. Of course, this effectively makes a^{-n} the multiplicative inverse of a^n and thus is similar reasoning to using the law of exponents, and it would require further alteration to extend to rational exponents. Reflection on the meaning of exponents, even for natural numbers, played almost no role in the classroom discussion. The slight exception (not quoted above) was the teacher rejecting a student suggestion that $2^{-4} = 2(-2)(-2)(-2)$ [sic] because it is not multiplying 2 by itself.

One might ask, in Sfard's episodes as well as our own, whether the teachers' reliance on patterns is an instance of RPG or of PPG? As in PPG, the teachers' arguments do emphasize the process generating the pattern and not merely the empirical results of a few examples, but we hesitate to call their presentations deductive. In fact, the question is based on a false premise. Recall that RPG and PPG are both classified as proof schemes: they are intended to address the

Need for Certainty. That is, there should be a well-defined mathematical question to answer, and a conjecture about that answer to validate. The question itself specifies the process intended to generate all the examples that should fit the conjectured pattern. For example, asking for the sum of the first n consecutive odd natural numbers specifies the process that generates every specific instance of this question. The conjectural answer, n^2, could be justified by PPG reasoning amounting to mathematical induction (perhaps informal). However, in Sfard's episodes and ours, the issue (from our perspective) is not one of achieving *certainty* regarding a conjecture, but rather necessitating the adoption of a mathematical convention[10]. This is not the intended context for RPG or PPG reasoning. What is lacking in particular is a good reason, grounded in a well-defined question being asked, for continuing the pattern.

5. Comparison of Episodes

The two episodes share many characteristics regarding the mathematical content, the pedagogical strategies, and the behavior of the teachers and the students. First, the mathematical content of each episode is the development of rules for extending mathematical operations from natural numbers to integers. Such rules are mathematical *conventions*, not theorems. For Sfard this is a meta-level task, not object level. In DNR, such knowledge would not arise from the Need for Certainty but rather from the needs for Communication, for Connection and Structure, and possibly for Causality or Computation. Neither teacher seems to be aware that this content is a convention and not a theorem. Sfard's teacher says, as previously quoted,

[10] Some treatments take the relevant properties as formal axioms and prove the rules for products or exponents with negative numbers (see below for an example), but neither of these teachers did so.

T: I want to explain what Sophie said. What she said is *true* … and this is the *right* answer. [Italics added.]

In her notes following the lesson she wrote, "I can see that even my repeated emphasis on the *correct* proposal did not help." And Sfard writes (p. 588), "The teacher hoped, however, that the explicit confrontation between the two alternatives would soon lead the class to the *unequivocal decision* about the preferability of Sophie's proposal." Our teacher, like Sfard's, uses the language of correctness or truth rather than agreement on a convention: "*figure out* the rest of the numbers", "this is the *evidence*, how can they argue with that?". The strategies of presentation and justification adopted by both teachers are surely influenced by their beliefs that they are teaching theorems.

It is essential that teachers be able to epistemologically distinguish mathematical conventions from theorems. Such a distinction is necessary to be able to focus on changes in the meta-rules of discourse and justification. The curriculum contains many other examples of content that is conventional in nature, for example, the "PEMDAS" rules for the order of operations, or the rules for writing radical expressions in simplest or reduced form. Levenson (2012) studied teachers' awareness of the distinction between definitions (conventional) and theorems (provable) in the context of zero exponents. She interviewed three experienced junior high school teachers in Israel (none of whom majored in mathematics). All three stated that $a^0 = 1$ is a theorem, and only one immediately saw $a^n = a \times a \times \ldots \times a$ (n times) as a definition. Moreover, only one was sure that definitions could not be proved (and another insisted that some definitions could be proved). Although Levenson's sample size is also small, this suggests that distinguishing definitions from theorems is problematic for many teachers.

Second, both teachers assumed that their students could easily determine and agree upon the "correct" rule. Sfard's teacher seemed confident that the "evidence" provided to the students was so compelling as to allow only one outcome, while our teacher took the halving pattern and the graphical evidence to be incontrovertible.

Third, both teachers used patterns as the primary justification they presented. This may reflect the familiarity of Result Pattern Generalization in students' prior classroom experience. Elementary mathematics curricula often include a large number of pattern recognition or discovery exercises. Despite contributing to students' number sense, awareness of patterns, and ability to make conjectures, they can foster the undesirable belief that merely observing a pattern in a limited number of examples entails that the pattern is correct and unique. Both textbooks and teachers may implicitly endorse the empirical proof scheme by relying on patterns for justification. As illustrated above, in both episodes the students suggested several other ways of generalizing the results, demonstrating the limits of RPG reasoning.

It is instructive to expand on how the patterns used by the teachers embody the relevant mathematical properties, namely the distributive property in Sfard's case and the laws of exponents in our own. In Sfard's case, the distributive property $a(b+c)=ab+ac$ is the fundamental axiomatic link between addition and multiplication. The special cases $a(b\pm 1)=ab\pm a$ contain the essential information when the variables are integers. Suppose that addition of signed numbers has been defined already, but multiplication has not. Then for consistency with the distributive property the product *ab* should be defined in such a way that increasing (respectively, decreasing) *b* by one unit increases (respectively, decreases) the product by *a*. Starting from a base case such as $a\times 0=0$, this determines the "correct" values of all products. In the case of positive integers, the definition of multiplication as repeated addition embodies this requirement, but a more general viewpoint is needed for negative integers. This is precisely what the patterns used by Sfard's teacher accomplish. However, they are not explained in these terms, and the teacher may not be aware of the connection. In response to a query like Sophie's about needing to extend the pattern to find a large product, a more direct explanation could be emphasized. Suppose *a* and *b* are positive integers. To find $a \times (-b)$ using the distributive property, one can notice that $0 = a \times (b + (-b)) = a \times b + a \times (-b)$, so that $a \times (-b) = -(a \times b)$. With this established, a similar argument finds $(-a) \times (-b)$.

In our case of negative exponents, the key property is $2^{a+b} = 2^a \times 2^b$, or the special case $2^{a+1} = 2^a \times 2$. For positive exponents this property is ensured by the definition that exponentiation is repeated multiplication[11]. More generally, for integers, it requires that 2^a be defined in such a way that increasing (respectively, decreasing) *a* by one unit multiplies (respectively, divides) the result by 2. This was exactly the pattern that our teacher directed the students to employ. The two contexts are thus precisely analogous[12].

[11] In fact, it is a theorem for natural number exponents.

[12] Once again the general case could be found at once by seeing that $1 = 2^{b+(-b)} = (2^b)(2^{-b})$, so $2^{-b} = \frac{1}{2^b}$.

Fourth, neither teacher made explicit the underlying axiom that guided the change in discourse: the desire to preserve the distributive property in the multiplication episode, or the laws of exponents in the other episode. These axioms provide a reason—separate from the teacher's authority—for extending definitions in particular ways. Such axioms often appear in elementary curricula simply as names for obvious facts about numbers, which students must know but that play no particular role in justification or problem solving. The teachers presumably know these axioms by name but may not be explicitly aware of how they underlie the patterns used or the conventions adopted. However, classroom discussions like those presented here can provide valuable opportunities for using these axioms in a nontrivial way and making their roles an explicit object of mathematical discussion. We are not suggesting that arithmetic should be taught in a formal deductive manner starting from such axioms, or that the term "axiom" should even be used. Rather, the axioms function as basic properties summarizing the classroom community's experience with numbers and quantities. Reflecting on their role can be a step toward a future understanding of the deductive structure of mathematics.

Fifth, neither teacher could convince the class that the pattern (s)he preferred was more correct than, or preferable to, the alternative proposed by students. The teachers were surprised by the fact that students remained unconvinced, and struggled to find an explanation that students would accept. It is possible that the teachers had trouble accepting that more than one pattern can fit a finite set of data or can be persuasive. As for the behavior of the students, in both episodes they proposed an alternative to the teacher's rule and clung to it tenaciously; the teacher's authority rather than his/her arguments ultimately settled the debate. The students' alternatives took the form of computational rules, and the students interpreted the teacher's attempt at justification as being a computational algorithm that did not make sense (instead of a pattern-based justification) and rejected it as such.

We conjecture that the students' objections were based on their Need for Causality or for Computation. That is, they needed to know *why* the outcome of a process of exponentiation or multiplication was a fraction, or had the claimed sign, and this need was not addressed by the existence of a pattern that the result fit into. They expected a computational recipe that incorporated appropriate operations to address this need.

6. Pedagogical Implementation of the Necessity Principle

The major didactical obstacle common to both episodes arose from the teachers presenting a situation requiring the adoption of a convention, but framing the task for students differently: as one of determining a provably correct answer. This led to confusion and resistance on the students' part. The DNR framework suggests alternative pedagogical treatments that might avoid this obstacle. One possible way to avoid student objections, such as those voiced in these episodes, is to help them feel intellectual need for the desired content (or some framework that would help settle the debate). The Necessity Principle requires that instruction address an intellectual need in order for students to learn. Intellectual need is not "one size fits all": what constitutes intellectual need for a student depends on that student's prior knowledge, beliefs, mathematical sophistication, and so forth (Harel, 2013). We have already observed that the definitions of arithmetic operations on negative numbers are conventions rather than theorems. However, there are rational reasons to adopt conventions in general and these conventions in particular. In this section we suggest some alternate instructional treatments that have the potential to address the intellectual need of the students in Sfard's episode and our own. In general, these may be of two types: those that rely on a physical model for the operations, and those that deal with intrinsically mathematical criteria. We expect that reliance on models will gradually decrease as students progress: for younger students anything called "multiplication" should prove its usefulness in typical multiplicative situations, while for advanced students the desire to preserve certain fundamental properties may be sufficient.

Since the students in both studies seem to have a Need for Causality/Computation, we suggest addressing this directly by repackaging the pattern-based evidence explicitly as a computational method. Example problem: Mary suggested computing $2 \times (-3)$ as $2 \times (1 - 4) = 2 - 8 = -6$. John used the same idea but computed $2 \times (4 - 7) = 8 - 14 = -6$. Is it coincidental that they obtained the same answer? Will all students using this idea obtain the same answer? If so, what causes this agreement?

Using the distributive property[13] as a computational tool may be natural for students because of familiar decimal algorithms like $2 \times 26 = 2 \times (20 + 6) = 40 + 12 = 52$. Using it in this way to "compute" signed products involves the same ideas Sfard's teacher appealed to but has several advantages. It is explicitly computational, requiring only multiplication of positive numbers. It focuses attention on the distributive property rather than keeping it hidden, and it sets up a puzzle that requires reasoning from the students rather than mere pattern recognition. Students might settle on the "purest" form of the idea, computing $2 \times (-3)$ as $2 \times (0 - 3)$, as

[13] Technically, the hypothetical students are using the distributive property over subtraction rather than addition, which could lead to its own discussion.

easiest to use. Once students show that such computations do yield a well-defined answer, one can change viewpoints and propose that $2 \times (-3)$ be *defined* as the common result of all such computations. This approach shows clearly why $2 \times (1 - 4)$ must have the opposite sign from $2 \times (4 - 1)$. Students can extend their reasoning to explore whether the distributive property will continue to hold when applied to expressions explicitly containing negative numbers, e.g. $(-2) \times (5 - 3)$. They can then use it similarly to define products of two negative factors, such as $(-3) \times (-2)$. Such formal computations, later verified to give well-defined results, often lead to the creation of new mathematics, as in the historical cases of negative and complex numbers. For example, Cardano's formula for solving a cubic equation produces expressions involving complex numbers even when the roots are real. This provided an intellectual need to define those expressions in such a way as to agree with the known real roots.

We emphasize that this approach is not a deductive proof of the rules for multiplying signed numbers. It is an exploration of whether the use of the distributive property for computation is well-defined in this extended context of signed numbers: will it produce unique results independent of the choices made by users? If so, then we have the option of adopting it as a convention for extending the meaning of "multiplication" to this context.

A teacher could also explore alternate proposals such as Roi's in depth to make their consequences explicit. Roi's rule of taking the sign of the number larger in magnitude leaves $3 \times (-3)$ undefined, violates the distributive property in examples like $3 \times [\ 2 + (-2)\ ]$, and would assign very different values to $2.99 \times (-3)$ and $3.01 \times (-3)$, which would make estimation of products problematic (formally, it has the consequence that a product is not a continuous function of its factors).

The Need for Communication is the most natural basis for adopting conventions, and it can provide reasons for adopting a new discourse. Here is a treatment of negative exponents from this point of view, based on a physical model. Two scientists, A and B, are studying the growth of bacteria in a Petri dish. The bacteria double in number every hour. Scientist A begins his observations at a time he designates as t=0 hours, with 1000 bacteria in his dish. His population after t hours will be $A(t) = 1000 \times 2^t$ bacteria. Scientist B enters the lab 5 hours later and begins observing the same dish at time t=5, which she calls T=0 using her own clock. She finds that the population fits the function $B(T) = 32000 \times 2^T$. Question: how should scientist B describe the observations made prior to her arrival, by scientist A? For example, how does she express the fact that the population 3 hours before she arrived was 4000?

This question creates a need to make sense of the natural answer, $B(-3) = 32000 \times 2^{-3} = 4000$, just as Cardano needed to make sense of “known” real values for expressions involving complex numbers. Of course, scientist B may insist that negative exponents are undefined, and may invent a completely new notation for the function giving the population in her past. However, writing $B(T) = 1000 \times 2^{T+5}$ may increase her comfort with substituting T= −3. Since nothing about the bacteria themselves changes at time T=0, why change the function describing them? We have a natural notation which has not yet been assigned a meaning, so why not assign it the natural meaning in this context? Why not take advantage of the opportunity for communication by observing that the substitution $t = T + 5$ makes both functions agree?

Of course, in the situation described, something about the bacteria *did* change at t=0 when Scientist A began the experiment. But nothing prevents us from imagining that Scientist A instead walked in on an experiment in progress, begun even earlier by someone else. In this way we create the abstraction of a process that continues in a uniform way into the indefinite past and future. The exponential function we are defining is intended to model such processes and therefore should be defined in a uniform way for all integer arguments. If the scientists consider that the bacteria grow at a uniform rate even for times shorter than an hour, so that during any time interval the population is multiplied by a factor depending only on the length of that time interval, they can continue their analysis to define rational powers of 2 as well. (This will make more sense if the absolute numbers are much larger and fractions of a bacterium are ignored.) For example, $2^{1/2}$ is the factor by which the population increases in half an hour, and must equal $\sqrt{2}$ so that the population doubles in two consecutive half hours.

The analogous experiments in Sfard's situation of signed-number multiplication would involve the distance-rate-time formula $d = rt$. The two scientists observe the positions of an object moving along a straight track at a constant speed, but one begins her observations earlier than the other. For example, Scientist A observes the object at position $d = 2t$ at time t, while Scientist B, arriving five minutes later, observes it at $d = 2T + 10$ when her own watch reads T. They can also allow the rate r to have either sign in order to distinguish the two directions of travel along the track.

In terms of our earlier question of whether pattern-based arguments for mathematical conventions are examples of RPG or PPG, the function of the physical model in these context-based teaching strategies is to provide a good reason (in terms of the model) for a specific pattern. The convention would then be to apply the general pattern uniformly, so that the same rules work outside the context of the physical model and for any numbers considered (within the new domain).

7. Conclusion: Contrasting the DNR Analysis with the Commognitive Analysis

Although a commognitive analysis leads to some of the same observations about these episodes as the DNR analysis, we think that considering the Necessity Principle as a lens for analysis yields additional benefits. Specifically, it allows further analysis of the students' reactions to their teachers' presentations and suggests concrete pedagogical recommendations. We think there is additional nuance to the following statements of Sfard's (2007).

"According to commognitive analysis, learning about negative numbers involves a transition to a new, incommensurable discourse" (p. 597).

"All the parties to the learning process need to agree to live with the fact that the new discourse will initially be seen by the participating students as somehow foreign, and that it will be practiced only because of its being a discourse that others use and appreciate" (p. 609).

“This process of thoughtful imitation seems to be the most natural way, indeed the only imaginable way, to enter into new discourses. It is driven by the need to communicate [In this context, DNR would consider this a social rather than intellectual need]… The learners accept a rule enacted by another interlocutor as a prelude to, rather than a result of, their attempts to figure out the inner logic of this interlocutor’s discourse” (p.610).

A simplistic reading of these statements might suggest that students cannot initially learn these kinds of ideas in a meaningful way; rather, they must go along with the teacher’s ideas despite not understanding the logic of these, and the logic will come later. We would disagree with this contention, and we believe that the more subtle issue in the episodes is how the nature of the justification or activities is appropriated by students. For this, we agree with Sfard that the students may need to go along with the teacher’s suggested activities or means of justification before fully understanding these. For instance, consider our example of whether computations like $2 \times (1 - 4) = 2 - 8 = -6$ and $2 \times (4 - 7) = 8 - 14 = -6$ must always agree. Students may not be used to comparing different computations and considering why they will or will not yield the same result. However, we would say that these ideas are fairly natural in that students can be brought to appreciate them within a relatively short time (potentially during the same class period they are first introduced).

We think that the Necessity Principle allows for concrete improvements in the instructional approaches used in these episodes. It is true that a student confronted with a new type of discourse cannot be immediately aware of all the implications of adopting that new discourse. Mathematicians who introduce new concepts and methods also do not initially see all their ramifications. (Indeed, the ramifications of even simple mathematical ideas are likely inexhaustible and still being uncovered.) However, it does not follow that adopting a new discourse must always be a pure leap of faith. The original creator of the new discourse had intellectual reasons for creating it, and obviously did not adopt it from another person. One can bring students to see the need being addressed and enough advantages of the new ideas to make it rational to pursue them further. Teachers, as expert guides or translators, should be fluent in both the students' "old" discourse and the "new" discourse of the mathematical community. They should introduce the new discourse only after students have an intellectual need for it. To do so effectively, they should know what has necessitated new mathematical discourses historically, and what necessitates them pedagogically.

The episodes examined illustrated both epistemological obstacles—such as the unavoidable issue of not being able to naturally interpret a product of two negative numbers in terms of equal groups of objects—and didactical obstacles—such as both teachers' insistence that there was a "right answer". We have argued that (1) these kinds of situations can provide rich opportunities for student reasoning (taking advantage of the epistemological obstacles, while minimizing the didactical obstacles), and (2) DNR-based instruction provides concrete suggestions for improving the teaching of these episodes by providing necessity for the conventions adopted, in contrast to the commognitive framework in which the expectation is that students adopt a new discourse even though it is foreign to them and they will only later understand why it works.

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