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$\mathcal{D}$-modules on 1|1 supercurves

Jeffrey M. Rabin [a], Mitchell J. Rothstein [b,*]

[a] Department of Mathematics, UCSD, La Jolla, CA 92093, United States
[b] Department of Mathematics, University of Georgia, Athens, GA 30602, United States

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ABSTRACT

It is known that to every (1|1)-dimensional supercurve X there is associated a dual supercurve $\hat{X}$, and a superdiagonal $\Delta \subset X \times \hat{X}$. We establish that the categories of $\mathcal{D}$-modules on X, $\hat{X}$ and Δ are equivalent. This follows from a more general result about $\mathcal{D}$-modules and purely odd submersions. The equivalences preserve tensor products, and take vector bundles to vector bundles. Line bundles with connection are studied, and examples are given where X is a super elliptic curve.

1. Introduction

Super Riemann surfaces, the analogue of Riemann surfaces in the category of supermanifolds, have been extensively studied since their introduction in the context of superstring theory [1–4]. They are supercurves of dimension (1|1) satisfying an additional "superconformal" constraint which allows one to identify irreducible (Weil) divisors with points. The study of more general supercurves X, without the constraint, began in earnest with the paper [5], where it was observed that the irreducible divisors on X can be identified with points of a dual supercurve, $\hat{X}$, having the same underlying topological space, and that the dual of $\hat{X}$ is again X. Furthermore, there is a distinguished (1|2)-dimensional submanifold,

$$\Delta \subset X \times \hat{X}$$

(the "superdiagonal"), which exhibits $\hat{X}$ as the family of irreducible divisors on X and vice versa. Supercurves, and their duality, were later seen in [6] to play an important role in the study of super analogues of the KP-hierarchy. In that paper the main objects of study were $\mathrm{Pic}(X)$ and $\mathrm{Pic}(\hat{X})$.

In this paper we explore the categories of $\mathcal{D}$-modules over X, $\hat{X}$ and Δ. We find in Section 4 that in fact the three categories $\mathcal{D}_X$-mod, $\mathcal{D}_{\hat{X}}$-mod and $\mathcal{D}_\Delta$-mod are equivalent. This follows from a more general statement in Section 2, about $\mathcal{D}$-modules and purely odd submersions. Section 3 provides a review of supercurve duality. The equivalence of categories is further explored in Sections 5–7, where explicit formulas in local coordinates are given. Section 8 describes the direct image of a trivial vector bundle with connection, and Section 9 specializes this to line bundles. Section 10 illustrates our results with explicit examples for the case of super elliptic curves, i.e., supercurves of genus one.

* Corresponding author.
E-mail addresses: jrabin@math.ucsd.edu (J.M. Rabin), rothstei@math.uga.edu (M.J. Rothstein).

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Notice that our setup involves a double fibration,

$$\begin{array}{ccc} & \Delta & \\ \pi \swarrow & & \searrow \hat{\pi} \\ X & & \hat{X} \end{array} \tag{1.1}$$

reminiscent of the Fourier–Mukai transform [7]. The situation is simpler, however, in that the categories of $\mathcal{D}$-modules on all three spaces are equivalent.

Throughout the paper we work in the category of supermanifolds over a fixed superscheme S over $\mathbb{C}$. By supermanifold we mean a smooth morphism $Z \to S$, and by dimension we mean the relative dimension. Though most of the results in this paper are valid for arbitrary S, we will avoid notational complications by assuming that $S = \mathrm{Spec}(\Lambda)$, where Λ is a finite-dimensional nilpotent extension of $\mathbb{C}$. Except as noted, the results are valid in both the Zariski and complex topologies.

2. $\mathcal{D}$-modules and purely odd submersions

Let $\sigma : Z \to W$ be a smooth morphism of smooth superschemes over S. Let $\mathcal{D}_Z$ denote the sheaf of linear differential operators on $\mathcal{O}_Z$. Then a $\mathcal{D}_Z$-module is a sheaf $\mathcal{F}$ of $\mathcal{O}_Z$-modules equipped with a flat connection

$$\nabla : \Omega_Z \otimes \mathcal{F} \to \Omega_Z[1] \otimes \mathcal{F}$$

where Ω_Z is the sheaf of differential one-forms relative to S, and $\Omega_Z[1]$ denotes Ω_Z with a degree shift. (For background on $\mathcal{D}$-modules, see [8,9].) One has direct and inverse image functors for $\mathcal{D}$-modules. The inverse image of a $\mathcal{D}_W$-module $\mathcal{F}$ has $\sigma^*(\mathcal{F})$ as its underlying $\mathcal{O}_Z$-module. The direct image is, in general, defined in the derived category.

Assume now that σ is a submersion. Then there is an underived version of the direct image, defined as follows. Let $\mathcal{T}_Z$ denote the tangent sheaf of Z and let $\mathcal{T}_\sigma \subset \mathcal{T}_Z$ denote the vertical tangent sheaf. Then we have an exact sequence

$$0 \to \mathcal{T}_\sigma \to \mathcal{T}_Z \to \sigma^*(\mathcal{T}_W) \to 0.$$

The direct image functor

$$\sigma_+ : \mathcal{D}_Z\text{-mod} \to \mathcal{D}_W\text{-mod}$$

is defined by

$$\sigma_+(\mathcal{F}) = \sigma_*(\mathrm{ann}_{\mathcal{F}}(\mathcal{T}_\sigma)) \tag{2.1}$$

where $\mathrm{ann}_{\mathcal{F}}(\mathcal{T}_\sigma)$ denotes the subsheaf of $\mathcal{F}$ annihilated by $\mathcal{T}_\sigma$.

Say that the submersion $\sigma : Z \to W$ is *purely odd* if the fibers have dimension $(0|n)$ for some n. The key observation in the paper is the following result.

Theorem 2.1. *Let $\sigma : Z \to W$ be a purely odd submersion. Then the categories of $\mathcal{D}$-modules on Z and W are equivalent. Specifically, the functors σ^* and σ_+ are inverse equivalences.*

Proof. To simplify the notation, note that Z and W share the same underlying topological space. If $\mathcal{G}$ is a $\mathcal{D}_W$-module, then $\mathcal{G}$ maps naturally to $\sigma^*(\mathcal{G})$. A computation in local coordinates easily shows that this map is injective, and that the image of $\mathcal{G}$ is precisely $\sigma_+\sigma^*(\mathcal{G})$.

If $\mathcal{F}$ is a $\mathcal{D}_Z$-module, then $\sigma_+(\mathcal{F}) \subset \mathcal{F}$, and we have a natural map

$$\mathcal{O}_Z \otimes_{\mathcal{O}_W} \sigma_+(\mathcal{F}) = \sigma^*\sigma_+(\mathcal{F}) \to \mathcal{F}.$$

The fact that this map is an isomorphism follows from the purely algebraic lemma stated below. □

Lemma 2.2. *Let $R = R_0 \oplus R_1$ be a $\mathbb{Z}_2$-graded ring. Let*

$$Q = R[\theta_1, \ldots, \theta_n, \partial_1, \ldots, \partial_n]$$

where the θ_i's are free supercommuting odd variables, $\partial_i = \partial/\partial\theta_i$, and the θ's and ∂'s supercommute with R. Then the categories Q-mod and R-mod are equivalent. Specifically, the following functors are inverses:

$$Q\text{-}mod \ni M \mapsto M_* = \mathrm{ann}(\partial_1, \ldots, \partial_n) \tag{2.2}$$

$$R\text{-}mod \ni N \mapsto N^* = R[\theta_1, \ldots, \theta_n] \otimes_R N. \tag{2.3}$$

The statement that the natural homomorphism

$$(M_*)^* \simeq M$$

is an isomorphism is equivalent to the statement that every $A \in M$ has a unique expansion

$$A = \sum_{\mu} A_{\mu}\theta_{\mu}, \quad A_{\mu} \in M_{*} \tag{2.4}$$

where $\mu = (\mu_1, \ldots, \mu_n)$ is a multiindex of 0's and 1's.

Proof. It is easy to see that for all R-modules N, $N \simeq (N^{*})_{*}$.

For the other direction, one has the natural map

$$R[\theta_1, \ldots, \theta_n] \otimes_R M_{*} \to M \tag{2.5}$$

which is an isomorphism if and only if formula (2.4) is valid. We prove (2.4) by induction on n. For $n = 1$, let $A \in M$. Then the decomposition is given by

$$A_0 = A - \theta_1\partial_1 A \tag{2.6}$$

$$A_1 = \partial_1 A. \tag{2.7}$$

For the uniqueness, apply ∂_1 to both sides of the equation

$$A = A_0 + \theta_1 A_1.$$

If $n > 1$, write

$$R[\theta_1, \ldots, \theta_n, \partial_1, \ldots, \partial_n] = R[\theta_2, \ldots, \theta_n, \partial_2, \ldots, \partial_n][\theta_1, \partial_1]. \quad \square$$

Proposition 2.3. *Let $\sigma : Z \to W$ be a purely odd submersion. Then σ_+ is exact, and preserves tensor products. It takes $\mathcal{D}_Z$-modules that are locally free as $\mathcal{O}_Z$-modules to locally free $\mathcal{O}_W$-modules, and preserves rank.*

Proof. The first part follows immediately from Theorem 2.1. Since the proposition is local, we may assume for the second part that we have a flat connection on the trivial bundle of rank $p|q$, i.e., $\mathcal{O}_Z^{p|q}$. Let the fiber dimension of σ be $0|n$, and let $\theta_1, \ldots, \theta_n$ be a set of fiber coordinates. Let $I \in \mathfrak{gl}^{p|q}(\mathcal{O}_Z)$ denote the identity matrix. Then we have a unique decomposition

$$I = \sum_{\mu} A_{\mu}\theta_{\mu}$$

where μ is a multiindex, such that for all i, $\nabla_{\theta_i} A_{\mu} = 0$. Let $A = A_{(0,\ldots,0)}$. Then A is an invertible matrix. Thus the columns of A lie in $\sigma_+(\mathcal{O}_Z^{p|q})$ and form a basis for $\mathcal{O}_Z^{p|q}$. If $\psi \in \sigma_+(\mathcal{O}_Z^{p|q})$, then there is a unique vector $\phi \in \mathcal{O}_Z^{p|q}$ such that $\psi = A\phi$. Then $0 = \nabla_{\theta_i}(A\phi) = A\partial_{\theta_i}\phi$, whence the entries of ϕ belong to $\mathcal{O}_W$. Thus the columns of A form a basis for $\sigma_+(\mathcal{O}_Z^{p|q})$ as an $\mathcal{O}_W$-module. $\square$

Remark 2.4. The standard result in the commutative setting, that a $\mathcal{D}$-module is locally free of finite rank as an $\mathcal{O}$-module if and only if it is coherent as an $\mathcal{O}$-module [8,9], holds in the supercommutative setting as well.

3. Supercurves and their duals

By definition, a supercurve is a supermanifold of dimension $(1|n)$ for some $n \geq 1$. Let X be a supercurve of dimension $(1|1)$. Then there is a dual $(1|1)$-dimensional supercurve $\hat{X}$ constructed as follows [5,6]. Define

$$\Delta_X = \mathrm{Proj}(\Omega_X) \xrightarrow{\pi} X. \tag{3.1}$$

Thus, if (z, θ) are local coordinates on an open set $\mathcal{U}$, $\Omega_X(\mathcal{U})$ is the polynomial algebra $\mathcal{O}_X(\mathcal{U})[\mathrm{d}\theta, \mathrm{d}z]$, with $\mathrm{d}\theta$ even and $\mathrm{d}z$ odd. In the proj construction, $\mathrm{d}\theta$, $\mathrm{d}z$ are taken as homogeneous coordinates, of which only $\mathrm{d}\theta$ may be inverted. Thus, on $\pi^{-1}(\mathcal{U})$ one has the local coordinate system

$$(z, \theta, \rho), \tag{3.2}$$

where

$$\rho = \mathrm{d}\theta^{-1}\mathrm{d}z. \tag{3.3}$$

In particular, Δ_X has relative dimension $(1|2)$ over S.

The exterior derivative d is an odd derivation of degree one:

$$\mathrm{d} : \Omega_X \to \Omega_X[1].$$

We may localize d, yielding an odd derivation

$$\tilde{\mathrm{d}} : \mathcal{O}_{\Delta_X} \to \mathcal{O}_{\Delta_X}(1), \tag{3.4}$$

where $\mathcal{O}_{\Delta_X}(1)$ is the twisting sheaf [10] associated to the graded Ω_X-module $\Omega_X[1]$. Note that $\mathrm{d}\theta$ is a trivialization of $\mathcal{O}_{\Delta_X}(1)$. Then we have the formula

$$\tilde{\mathrm{d}} = \mathrm{d}z\,\partial_z + \mathrm{d}\theta\partial_\theta = \mathrm{d}\theta(\rho\partial_z + \partial_\theta). \tag{3.5}$$

The *dual curve*, $\hat{X}$, has the same topological space as X, with structure sheaf

$$\mathcal{O}_{\hat{X}} = \ker(\tilde{\mathrm{d}}). \tag{3.6}$$

Let

$$u = z - \theta\rho.$$

Then $\tilde{\mathrm{d}}\rho = \tilde{\mathrm{d}}u = 0$. Furthermore, (u, ρ, θ) is a local coordinate system on Δ_X, and one checks that (u, ρ) is a local coordinate system on $\hat{X}$. In particular, $\hat{X}$ is a family of smooth $(1|1)$-dimensional supercurves over S. It is known [6] that Δ_X and $\Delta_{\hat{X}}$ are naturally isomorphic, as superschemes over $\hat{X}$. On the level of structure sheaves, the isomorphism is given in local coordinates by

$$\mathcal{O}_{\Delta_{\hat{X}}} \to \mathcal{O}_{\Delta_X}$$

$$u \mapsto z - \theta\rho \tag{3.7}$$

$$\rho \mapsto \rho \tag{3.8}$$

$$\mathrm{d}\rho^{-1}\mathrm{d}u \mapsto \theta. \tag{3.9}$$

Then $\mathcal{O}_{\hat{\hat{X}}}$ appears as a subsheaf of $\mathcal{O}_{\Delta_X}$, and one checks that this subsheaf coincides with the image of $\mathcal{O}_X$. Thus, $\hat{\hat{X}}$ is naturally isomorphic to X. In local coordinates, the isomorphism is given by

$$u - \rho\frac{\mathrm{d}u}{\mathrm{d}\rho} \mapsto z \tag{3.10}$$

$$\mathrm{d}\rho^{-1}\mathrm{d}u \mapsto \theta. \tag{3.11}$$

One should view $\Delta_X \cong \Delta_{\hat{X}}$ (denoted simply Δ in the sequel) as the "superdiagonal" in $X \times \hat{X}$ defined by the equation $z - u - \theta\rho = 0$.

4. Equivalences of categories

Theorem 4.1. *The categories $\mathcal{D}_X$-mod, $\mathcal{D}_{\hat{X}}$-mod and $\mathcal{D}_\Delta$-mod are equivalent.*

Proof. The maps $\pi : \Delta \to X$ and $\hat{\pi} : \Delta \to \hat{X}$ are purely odd submersions, so the result follows from Theorem 2.1. □

For $\mathcal{F}$ a $\mathcal{D}_X$-module, define

$$\hat{\mathcal{F}} = \hat{\pi}_+\pi^*(\mathcal{F}). \tag{4.1}$$

Then we have a canonical isomorphism $\hat{\hat{\mathcal{F}}} \simeq F$, by Theorem 4.1.

Example 4.2. By definition, $\hat{\mathcal{O}}_X = \mathcal{O}_{\hat{X}}$.

Example 4.3. Consider $\mathcal{D}_X$ as a left $\mathcal{D}_X$-module. We have

$$\pi^*(\mathcal{D}_X) = \mathrm{Diff}(\mathcal{O}_X, \mathcal{O}_\Delta).$$

Then a germ $L \in \mathrm{Diff}(\mathcal{O}_X, \mathcal{O}_\Delta)$ belongs to $\hat{\mathcal{D}}_X$ if and only if

$$\tilde{\mathrm{d}} \circ L = 0$$

where $\tilde{\mathrm{d}}$ is as in (3.4). We therefore have

$$\hat{\mathcal{D}}_X = \mathrm{Diff}(\mathcal{O}_X, \mathcal{O}_{\hat{X}})$$

which is to say, differential operators from $\mathcal{O}_X$ to $\mathcal{O}_\Delta$ that factor through the inclusion $\mathcal{O}_{\hat{X}} \to \mathcal{O}_\Delta$.

5. Local description

Let $\mathcal{U} \subset X$ be an open set, with coordinates (z, θ). Let (u, ρ) be the corresponding coordinates on $\hat{X}$. Then we get an isomorphism

$$\mathcal{O}_X|_{\mathcal{U}} \xrightarrow{\Psi^{(z,\theta)}} \mathcal{O}_{\hat{X}}|_{\mathcal{U}} \tag{5.1}$$

sending $z \to u$ and $\theta \to \rho$. That is,

$$\Psi^{(z,\theta)}(f(z) + \theta g(z)) = f(z - \theta\rho) + \rho g(z - \theta\rho) \tag{5.2}$$

$$= f(z) + \rho(\theta\partial_z f + g). \tag{5.3}$$

The isomorphism $\Psi^{(z,\theta)}$ extends to an isomorphism

$$\Psi^{(z,\theta)} : \mathcal{D}_X|_{\mathcal{U}} \xrightarrow{\sim} \mathcal{D}_{\hat{X}}|_{\mathcal{U}}$$

sending $\partial_z \mapsto \partial_u$ and $\partial_\theta \mapsto \partial_\rho$. This identification allows one to regard every $\mathcal{D}_X|_{\mathcal{U}}$-module as a $\mathcal{D}_{\hat{X}}|_{\mathcal{U}}$-module (in a coordinate-dependent way). If $\mathcal{F}$ is a $\mathcal{D}_X|_{\mathcal{U}}$-module, let $\mathcal{F}^{(z,\theta)}$ denote $\mathcal{F}$ itself, regarded as a $\mathcal{D}_{\hat{X}}|_{\mathcal{U}}$-module. Notice that Eq. (5.3) can be written

$$\Psi^{(z,\theta)}(h) = (1 - \theta\partial_\theta + \rho(\theta\partial_z + \partial_\theta))(h) \tag{5.4}$$

where $h = f + \theta g$. This suggests the following definition.

Definition 5.1. Let

$$\tau^{(z,\theta)} = 1 - \theta\nabla_\theta + \rho(\theta\nabla_z + \nabla_\theta). \tag{5.5}$$

(Here it is understood that one has a $\mathcal{D}_X|_{\mathcal{U}}$-module $\mathcal{F}$, and $\tau^{(z,\theta)}$ acts on $\pi^*\mathcal{F}$.)

Theorem 5.2. *Let $\mathcal{F}$ be a $\mathcal{D}_X|_{\mathcal{U}}$-module. Then $\tau^{(z,\theta)}$ gives an isomorphism*

$$\tau^{(z,\theta)} : \mathcal{F}^{(z,\theta)} \xrightarrow{\sim} \hat{\mathcal{F}}. \tag{5.6}$$

Proof. Let us establish first that $\tau^{(z,\theta)}$ gives a bijection $\mathcal{F} \to \hat{\mathcal{F}}$. In (z, θ, ρ) coordinates, $\mathcal{T}_{\hat{\pi}}$ is generated by

$$\mathbf{D} = \rho\partial_z + \partial_\theta. \tag{5.7}$$

Sections of $\pi^*\mathcal{F}$ may be uniquely written in the form $\phi = P + \rho Q$, where P and Q are sections of $\mathcal{F}$. Then $\phi \in \hat{\mathcal{F}}$ if and only if $\nabla_{\mathbf{D}}\phi = 0$, that is,

$$\nabla_\theta P + \rho(\nabla_z P - \nabla_\theta Q) = 0. \tag{5.8}$$

Set $P = P_0 + \theta P_1$ and $Q = Q_0 + \theta Q_1$, where ∇_θ annihilates P_i and Q_i. Then Eq. (5.8) reads

$$P_1 + \rho(\nabla_z P_0 + \theta\nabla_z P_1 - Q_1) = 0. \tag{5.9}$$

That is,

$$P_1 = 0, \qquad \nabla_z P_0 = Q_1. \tag{5.10}$$

Then

$$\phi = P_0 + \rho(Q_0 + \theta\nabla_z P_0) = \tau^{(z,\theta)}(P_0 + \theta Q_0). \tag{5.11}$$

Since $\phi = 0$ if and only if $P_0 = Q_0 = 0$, $\tau^{(z,\theta)}$ is bijective.

It remains to prove that for all $M \in \mathcal{D}_X|_{\mathcal{U}}$ and all $A \in \mathcal{F}$,

$$\tau^{(z,\theta)}(MA) = \Psi^{(z,\theta)}(M)\tau^{(z,\theta)}(A). \tag{5.12}$$

We leave it to the reader to check this when $M \in \mathcal{O}_X$. It remains to check Eq. (5.12) when M is a partial derivative. To distinguish between partial derivatives in the (z, θ, ρ) coordinate system and the the (u, ρ, θ) coordinate system, we will denote the latter by $\hat{\partial}_u$, etc. We also write $\hat{\nabla}_u$, etc. We then have

$$\hat{\nabla}_u = \nabla_z \tag{5.13}$$

$$\hat{\nabla}_\rho = \nabla_\rho - \theta\nabla_z \tag{5.14}$$

$$\hat{\nabla}_\theta = \nabla_\theta - \rho\nabla_z. \tag{5.15}$$

(These operators are acting on $\pi^*\mathcal{F}$.) Then

$$\hat{\nabla}_u \tau^{(z,\theta)} = \nabla_z \tau^{(z,\theta)} = \tau^{(z,\theta)} \nabla_z.$$

We also have

$$\nabla_\rho \tau^{(z,\theta)} = \tau^{(z,\theta)} \nabla_\rho + \nabla_\theta + \theta \nabla_z.$$

Then

$$\hat{\nabla}_\rho \tau^{(z,\theta)} = \tau^{(z,\theta)} \nabla_\rho + \nabla_\theta + \rho\theta \nabla_z (\nabla_\theta + \theta \nabla_z) = \tau^{(z,\theta)} \nabla_\rho + \tau^{(z,\theta)} \nabla_\theta.$$

The result follows, since one is applying this operator to the kernel of ∇_ρ. □

Theorem 5.2 provides a Čech description of the functor $\mathcal{F} \to \hat{\mathcal{F}}$. Cover X by coordinate charts $(\mathcal{U}_i, z_i, \theta_i)$. Then $\mathcal{O}_{\hat{X}}$ may be regarded as $\mathcal{O}_X$ glued to itself by transition automorphisms $D_{i,j} = \Psi^{(z_i,\theta_i)^{-1}} \Psi^{(z_j,\theta_j)}$. Then $D_{i,j}$ is a differential operator, so it can be applied to an arbitrary $\mathcal{D}_X$-module $\mathcal{F}$. Then Theorem 5.2 shows that $\hat{\mathcal{F}}$ is $\mathcal{F}$ glued to itself by the cocycle $D_{i,j}$.

6. Projected and injected supercurves

A *projected supercurve over* S, [6], is a submersion $\sigma : X \to X_0$ of smooth superschemes over S, where X and X_0 have relative dimension $(1|1)$ and $(1|0)$ over S, respectively. With the same assumptions on X and X_0, an *injected* supercurve over S is an immersion $\iota : X_0 \to X$.

Let $\sigma : X \to X_0$ be a projected supercurve. By Theorem 2.1, the categories $\mathcal{D}_X$-mod and $\mathcal{D}_{X_0}$-mod are equivalent. The following result is proved in [11]. For the reader's convenience we give a proof here.

Proposition 6.1. *Fix a smooth curve X_0/S. Then the category of projected supercurves $\sigma : X \to X_0$ is equivalent to the category of injected supercurves $\iota : X_0 \to X$. The equivalence is given by $X \mapsto \hat{X}$.*

Proof. Let $\sigma : X \to X_0$ be a projected supercurve. Then we have $\mathcal{O}_{X_0} \subset \mathcal{O}_X$. Let (z, θ) and (w, η) be two local coordinate systems on an open subset, such that $z, w \in \mathcal{O}_{X_0}$. Let $\rho = \mathrm{d}\theta^{-1}\mathrm{d}z$, $\lambda = \mathrm{d}\eta^{-1}\mathrm{d}w$. Writing $w = f(z)$ and $\eta = \theta g(z) + \Lambda(z)$, we have

$$\lambda = \frac{\rho f'(z)}{\rho(\theta g'(z) + \Lambda'(z)) + g(z)} \tag{6.1}$$

$$= \frac{\rho f'(z)}{g(z)}. \tag{6.2}$$

Thus we have a globally defined ideal $\mathcal{I} \subset \mathcal{O}_\Delta$, spanned locally by ρ. It is easily seen that we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_\Delta \xrightarrow{\alpha} \mathcal{O}_X \to 0.$$

Consider the restriction of α to $\mathcal{O}_{\hat{X}}$. Write a section of $\mathcal{O}_\Delta$ as $A + \rho B$, $A, B \in \mathcal{O}_X$. According to Eq. (5.8), $A + \rho B \in \mathcal{O}_{\hat{X}}$ if and only if $\partial_\theta A = 0$, $\partial_z A = \partial_\theta B$. In particular, $A \in \mathcal{O}_{X_0}$. Furthermore, for all $A \in \mathcal{O}_{X_0}$, we have $A + \rho\theta\partial_z A \in \mathcal{O}_{\hat{X}}$. Thus, α restricts to a surjection $\mathcal{O}_{\hat{X}} \to \mathcal{O}_{X_0}$, or equivalently, an immersion $X_0 \to \hat{X}$.

Conversely, let $\iota : X_0 \to X$ be an injected supercurve. Let $\mathcal{J}$ denote the kernel of the corresponding surjection $\iota^* : \mathcal{O}_X \to \mathcal{O}_{X_0}$. Let $\mathcal{U}$ be an open set sufficiently small that $\mathcal{J}|_\mathcal{U}$ is generated by an odd section θ. Then $\mathrm{d}\theta$ trivializes $\mathcal{O}_\Delta(1)|_\mathcal{U}$. We get a homomorphism of sheaves of rings,

$$\mathcal{O}_X|_\mathcal{U} \to \mathcal{O}_{\hat{X}}|_\mathcal{U} \tag{6.3}$$

$$f \mapsto \frac{\mathrm{d}(\theta f)}{\mathrm{d}\theta} = f - \theta \frac{\mathrm{d}f}{\mathrm{d}\theta}. \tag{6.4}$$

(This is the piece of $\tau^{(z,\theta)}$ that does not depend on a choice of z.) Let η be another generator of $\mathcal{J}|_\mathcal{U}$, and let $\eta = g\theta$, where $g \in \mathcal{O}_X^*(\mathcal{U})$. Then

$$\frac{\mathrm{d}(\eta f)}{\mathrm{d}\eta} = f - g\theta \frac{\mathrm{d}f}{g\mathrm{d}\theta - \theta \mathrm{d}g} \tag{6.5}$$

$$= f - g\theta \frac{\mathrm{d}f}{g\mathrm{d}\theta} = \frac{\mathrm{d}(\theta f)}{\mathrm{d}\theta}. \tag{6.6}$$

Thus we have a globally defined homomorphism of sheaves of rings,

$$\nu : \mathcal{O}_X \to \mathcal{O}_{\hat{X}}. \tag{6.7}$$

If we now extend θ to a coordinate system (z, θ), then

$$\nu(f(z) + \theta g(z)) = f + \rho\theta\partial_z f.$$

Thus the kernel of ν is $\mathcal{J}$, and we have produced an injection

$$\mathcal{O}_{X_0} \to \mathcal{O}_{\hat{X}}. \quad \square$$

With X and X_0 as above, let $\sigma : X \to X_0$ be a projected supercurve with corresponding injected supercurve $\iota : X_0 \to \hat{X}$. We then have two functors $\mathcal{D}_X$-mod $\to \mathcal{D}_{X_0}$-mod,

$$\mathcal{F} \mapsto \iota^*(\hat{\mathcal{F}}) \tag{6.8}$$
$$\mathcal{F} \mapsto \sigma_+(\mathcal{F}). \tag{6.9}$$

Proposition 6.2. *The functors (6.8) and (6.9) are naturally isomorphic.*

Proof. Let $\mathcal{F}$ be a $\mathcal{D}_X$-module. Formula (6.4) can be used on $\mathcal{F}$, to give a globally defined map

$$\nu : \mathcal{F} \to \hat{\mathcal{F}} \tag{6.10}$$
$$\phi \mapsto \frac{\nabla(\theta\phi)}{\mathrm{d}\theta}. \tag{6.11}$$

Restrict ν to $\sigma_+(\mathcal{F})$ and follow it with the pullback map $\hat{\mathcal{F}} \to \iota^*(\hat{\mathcal{F}})$ to obtain the desired isomorphism. $\square$

7. Split supercurves

Continuing with X and X_0 as above, X is said to be *split* over X_0 if there is a locally free rank-one sheaf $\mathcal{L}$ of $\mathcal{O}_{X_0}$-modules such that the structure sheaf of X is $\mathcal{O}_{X_0} \oplus \mathcal{L}$. This is equivalent to the existence of both a submersion $\sigma : X \to X_0$ and an immersion $\iota : X_0 \to X$, such that $\sigma \circ \iota = id$. Then by Proposition 6.1, $\hat{X}$ is also split over X_0, with the line bundle in question being the Serre dual of the original. Then by Theorem 2.1, we can identify the categories $\mathcal{D}_X$-mod and $\mathcal{D}_{\hat{X}}$-mod with $\mathcal{D}_{X_0}$-mod.

Proposition 7.1. *Let X be split over X_0. Under the equivalences*

$$\mathcal{D}_{X_0}\text{-mod} \cong \mathcal{D}_X\text{-mod}, \qquad \mathcal{D}_{X_0}\text{-mod} \cong \mathcal{D}_{\hat{X}}\text{-mod}$$

the transform $\mathcal{F} \to \hat{\mathcal{F}}$ reduces to the identity functor on $\mathcal{D}_{X_0}$-mod.

Proof. Given the submersion $\sigma : X \to X_0$ and immersion $\iota : X_0 \to X$, it is easy to check that the two functors σ_+ and ι^* are naturally isomorphic. Let $\mathcal{G}$ be a $\mathcal{D}_{X_0}$-module and write $\mathcal{G} = \sigma_+(\mathcal{F})$. Then the functor we are considering sends $\mathcal{G}$ to $\sigma_+(\hat{\mathcal{F}})$. We have

$$\sigma_+(\hat{\mathcal{F}}) \cong \iota^*(\mathcal{F}) \cong \sigma_+(\mathcal{F}) \cong \mathcal{G}. \quad \square$$

8. Direct image of the trivial bundle with connection

Returning now to the purely odd submersion $\sigma : Z \to W$, consider a connection $\mathrm{d} + \omega$ on the trivial bundle $\mathcal{O}_Z^{p|q}$. Here ω is a one-form with values in $\mathfrak{gl}^{p|q}(\mathcal{O}_Z)$, satisfying the zero-curvature condition

$$\mathrm{d}\omega + \omega \wedge \omega = 0. \tag{8.1}$$

According to Proposition 2.3, $\sigma_+(\mathcal{O}_Z^{p|q}, d + \omega)$ is a locally free $\mathcal{O}_W$-module, of rank $p|q$. It is natural to ask for a description of this $\mathcal{O}_W$-module.

The one-form ω restricts to a relative flat connection form on the fibers of σ. Denote this restriction by ω_σ.

Let $\mathcal{S}^{p|q}$ denote the sheaf of flat connection forms on Z and let $\mathcal{S}_\sigma^{p|q}$ denote the sheaf of relative flat connection forms. Let d_σ denote the relative differential. The sheaf $Gl^{p|q}(\mathcal{O}_Z)$ maps to $\mathcal{S}_\sigma^{p|q}$ by

$$A \mapsto -\mathrm{d}_\sigma A \cdot A^{-1}. \tag{8.2}$$

For a superscheme Z, let $\mathcal{N}_Z \subset \mathcal{O}_Z$ denote the sheaf of nilpotents.

Proposition 8.1. *The sheaf of subgroups $I + \mathfrak{gl}^{p|q}(\mathcal{N}_Z)$ maps surjectively to $\mathcal{S}_\sigma^{p|q}$.*

Proof. Let I denote the $(p+q)\times(p+q)$ identity matrix. Let $\theta_1,\ldots,\theta_n$ be fiber coordinates on an open set $\mathcal{U}\subset Z$. Decompose I as in Eq. (2.4).

$$I=\sum_{\mu}\theta_\mu A_\mu$$

where $(d+\omega)_\sigma(A_\mu)=0$. In particular,

$$-(\mathrm{d}_\sigma A_0)A_0^{-1}=\omega_\sigma. \tag{8.3}$$

Furthermore, the proof of Lemma 2.2 shows that

$$A_0=\prod_{i=1}^{n}(1-\theta_i\nabla_{\theta_i})(I)\in I+\mathfrak{gl}^{p|q}(\mathcal{N}_Z).\quad\square$$

If we regard $\mathcal{S}_\sigma^{p|q}$ as a sheaf of pointed sets, where the 0 one-form is the distinguished point, then the kernel of the map (8.2) is the sheaf $Gl^{p|q}(\mathcal{O}_W)$. We therefore have a connecting homomorphism

$$H^0(Z,\mathcal{S}_\sigma)\to H^1(W,I+\mathfrak{gl}^{p|q}(\mathcal{N}_W)) \tag{8.4}$$

(Z and W share the same topological space.)

Corollary 8.2. *Let $\mathcal{F}$ be a $\mathcal{D}_Z$-module with underlying $\mathcal{O}_Z$-module $\mathcal{O}^{p|q}$ and connection one-form ω. Regarding $\sigma_+(\mathcal{F})$ simply as a vector bundle, its class belongs to $H^1(W,I+\mathfrak{gl}^{p|q}(\mathcal{N}_W))$, and that class is the image of ω_σ under the connecting homomorphism (8.4).*

Proof. With the notation as in Proposition 8.1, the columns of A_0 form a basis for $\sigma_+(\mathcal{F})|_{\mathcal{U}}$. The result then follows from Eq. (8.3). $\square$

9. Line bundles with connection

Let ω be an odd closed one-form on X. Here we apply the results of the previous section to the transform of the trivial bundle, $\mathcal{O}_X$, endowed with the connection $d+\omega$.

Let $\Omega^1_{X,cl}$ denote the sheaf of closed one-forms. Recall that the map $\tilde{\mathrm{d}}$ takes values in the sheaf $\mathcal{O}_\Delta(1)$. Furthermore, $\mathcal{O}_\Delta(1)=\Omega^1_X$.

Lemma 9.1. *The image of $\tilde{\mathrm{d}}$ lies in $\Omega^1_{X,cl}$.*

(The next lemma implies that it is in fact all of $\Omega^1_{X,cl}$.)

Proof. Let (z,θ) be local coordinates on a neighborhood $\mathcal{U}\subset X$. Let $f\in\mathcal{O}_\Delta(\mathcal{U})$. Then there is a one-form $\omega\in\Omega^1_X$ such that $f=\frac{\omega}{\mathrm{d}\theta}$. Write $\mathrm{d}\omega=\mathrm{d}\theta\wedge\alpha$, $\alpha\in\Omega^1_X$. Then $\tilde{\mathrm{d}}f=\alpha$. Furthermore, $0=\mathrm{d}^2\omega=\mathrm{d}\theta\wedge\mathrm{d}\alpha$. We can cancel $\mathrm{d}\theta$, so $\mathrm{d}\alpha=0$. $\square$

Lemma 9.2. *The map $\tilde{\mathrm{d}}:\mathcal{N}_\Delta\to\Omega^1_{X,cl}$ is surjective.*

Proof. Let $\alpha\in\Omega^1_{X,cl}$. Then $\theta\frac{\alpha}{\mathrm{d}\theta}\in\mathcal{N}_\Delta$ and $\tilde{\mathrm{d}}(\theta\frac{\alpha}{\mathrm{d}\theta})=\alpha$. $\square$

Remark 9.3. Lemmas 9.1 and 9.2 imply the slightly weaker statement that there is an exact sequence

$$0\to\mathcal{O}_{\hat{X}}\to\mathcal{O}_\Delta\xrightarrow{\tilde{\mathrm{d}}}\Omega^1_{X,cl}\to 0. \tag{9.1}$$

On the other hand, it is known [6] that the quotient of $\mathcal{O}_\Delta$ by $\mathcal{O}_{\hat{X}}$ is the Berezinian sheaf, $Ber_{\mathcal{O}_{\hat{X}}}$. We therefore have the corollary

Corollary 9.4. *$\Omega^1_{X,cl}\simeq Ber_{\mathcal{O}_{\hat{X}}}$ as $\mathcal{O}_{\hat{X}}$-modules.*

This result seems to be new in this generality, although it is known for super Riemann surfaces where $X=\hat{X}$ [12,4].

By Lemmas 9.1 and 9.2 we have an exact sequence

$$0\to(\mathcal{N}_{\hat{X}})_0\to(\mathcal{N}_\Delta)_0\xrightarrow{\tilde{\mathrm{d}}}(\Omega^1_{X,cl})_1\to 0. \tag{9.2}$$

Theorem 9.5. *Let ω be an odd, closed one-form on X. Let c_ω denote the image of ω in $H^1(\hat{X}, \mathcal{N}_{\hat{X}})_0$ under the connecting homomorphism. Regarding $\widehat{\mathcal{O}_X^\omega}$ simply as a line bundle, its class in $H^1(\hat{X}, \mathcal{O}_{\hat{X}}^*)$ is $\exp(c_\omega)$.*

Proof. Pull ω back to Δ, giving the line bundle with connection $\mathcal{O}_\Delta^{\pi^*\omega}$. Let (z, θ) be a local chart on a neighborhood $\mathcal{U} \subset X$. Following the prescription in the proof of Corollary 8.2, decompose the constant function $1 \in \mathcal{O}_\Delta$ as

$$1 = \phi_0 + \rho\phi_1$$

where $\hat{\nabla}_\theta(\phi_i) = 0$. Then $\widehat{\mathcal{O}_X^\omega}$ is trivialized on $\mathcal{U}$ by ϕ_0. We have $\hat{\partial}_\theta = \frac{1}{\mathrm{d}\theta}\tilde{\mathrm{d}}$. Then

$$\phi_0 = 1 - \theta\hat{\nabla}_\theta(1) \tag{9.3}$$

$$= 1 - \frac{\theta}{\mathrm{d}\theta}\nabla_\theta(1) = 1 - \frac{\theta\omega}{\mathrm{d}\theta} \tag{9.4}$$

$$= \exp\left(-\frac{\theta\omega}{\mathrm{d}\theta}\right). \tag{9.5}$$

Furthermore,

$$\tilde{\mathrm{d}}\left(\frac{\theta\omega}{\mathrm{d}\theta}\right) = \frac{\mathrm{d}(\theta\omega)}{\mathrm{d}\theta} = \omega. \quad \square \tag{9.6}$$

Let us also note the following special case of Corollary 8.2.

Theorem 9.6. *Let $\sigma : X \to X_0$ be a projected supercurve. Then*

1. *The sequence*

$$0 \to \mathcal{N}_{X_0} \to \mathcal{N}_X \xrightarrow{d} \Omega^1_{X/X_0,cl} \to 0 \tag{9.7}$$

   is exact, where d is the relative differential.
2. *Let ω be an odd, closed one-form on X. Let $\omega' \in \Omega^1_{X/X_0,cl}$ denote the image of ω under the natural map $\Omega^1_X \to \Omega^1_{X/X_0}$. Let c_ω denote the image of ω' in $H^1(X_0, \mathcal{N}_{X_0})_0$ under the connecting homomorphism. Then the class of $\sigma_+(\mathcal{O}_X^\omega)$ in $H^1(X_0, \mathcal{O}_{X_0}^*)$ is $\exp(c_\omega)$.*

Our final result along these lines is a refinement of Theorem 9.5 in the case that $\mathcal{O}_X^\omega$ is the pullback of the trivial bundle with connection on X_0.

Theorem 9.7. *Let $\sigma : X \to X_0$ be a projected supercurve. Let $\mathcal{I} \subset \mathcal{O}_{\hat{X}}$ denote the ideal sheaf of X_0 in $\hat{X}$ with respect to the corresponding imbedding, $\iota : X_0 \to \hat{X}$. Identify $\Omega^1_{X_0}$ as a subsheaf of Ω^1_X by pullback. Then*

1. *The sequence*

$$0 \to \mathcal{I} \to \mathcal{I}\mathcal{O}_X \xrightarrow{\tilde{\mathrm{d}}} \Omega^1_{X_0} \to 0 \tag{9.8}$$

   is exact.
2. *Let ω be an odd one-form on X_0. (Note that ω is necessarily closed for reasons of dimension.) Let c_ω denote the image of ω in $H^1(\hat{X}, \mathcal{I})_0$ under the connecting homomorphism. Then the class of $\widehat{\mathcal{O}_X^\omega}$ in $H^1(\hat{X}, \mathcal{O}_{\hat{X}}^*)$ is $\exp(c_\omega)$.*

Proof. In (z, θ) coordinates on X, $\mathcal{I}$ is generated by ρ. Writing a section of $\mathcal{I}\mathcal{O}_X$ as $\rho g(z, \theta)$, we have $\tilde{\mathrm{d}}(\rho g) = \mathrm{d}z\frac{\partial g}{\partial \theta}$, which shows that the image of $\mathcal{I}\mathcal{O}_X$ under $\tilde{\mathrm{d}}$ is precisely $\Omega^1_{X_0}$. The kernel is $(\mathcal{I}\mathcal{O}_X) \cap \mathcal{O}_{\hat{X}}$, which must be shown to coincide with $\mathcal{I}$. One inclusion is obvious. For the other inclusion, take $\rho g \in \mathcal{I}\mathcal{O}_X$. Then $\tilde{\mathrm{d}}(\rho g) = 0$ if and only if $\partial g/\partial\theta = 0$, which is to say $g \in \mathcal{O}_{X_0}$. Then $\tau^{(z,\theta)}(g) = g + \rho\theta\frac{\partial g}{\partial z}$. Thus $\rho g = \rho\tau^{(z,\theta)}(g) \in \mathcal{I}$. This completes the proof of statement 1.
Statement 2 follows as in Theorem 9.5. $\square$

9.1. Invertible sheaves in the complex topology

In this subsection, we work in the complex topology, where the Poincaré lemma is available. Then the group of invertible sheaves on X equipped with a flat connection is the group $H^1(X, \Lambda^*)$. This group sees only the topology of X, and is therefore canonically isomorphic to $H^1(\hat{X}, \Lambda^*)$. In this way, an invertible sheaf with connection on X induces an invertible sheaf with connection on $\hat{X}$, by taking the same (constant) transition functions.

Proposition 9.8. *The identification of invertible sheaves with connection on X and invertible sheaves with connection on $\hat{X}$ given by $\hat{\pi}_+\pi^*$ coincides with the identity map on $H^1(X, \Lambda^*)$.*

If $\sigma : X \to X_0$ is a projected supercurve, then the identification of invertible sheaves with connection on X and invertible sheaves with connection on X_0 given by σ_+ also coincides with the identity map on $H^1(X, \Lambda^*)$.

Proof. We are given an invertible sheaf $\mathcal{L}$ on X with local trivializations ϕ_i on open sets $\mathcal{U}_i$, such that there are constants $c_{i,j} \in \Lambda^*$, such that $\phi_i = c_{i,j}\phi_j$. The connection is then defined by the condition that $\nabla(\phi_i) = 0$. Letting (z_i, θ_i) be a coordinate system on $\mathcal{U}_i$, the local trivializations for $\hat{\mathcal{L}}$ are $\tau^{(z_i,\theta_i)}(\phi_i)$. By Eq. (5.5), $\tau^{(z_i,\theta_i)}(\phi_i) = \phi_i$, so the transition functions are identical.

The proof of the second assertion is the same. □

10. Super elliptic curves

We illustrate the results of the previous section with a simple but nontrivial set of examples: super elliptic curves, i.e., supercurves of genus one [13]. Let $\mathbb{C}^{1|1}$ be the trivial family of supercurves $\mathrm{Spec}(\Lambda[z, \theta])$ over $S = \mathrm{Spec}(\Lambda)$. Let X be the quotient of $\mathbb{C}^{1|1}$ by the discrete group $G \cong \mathbb{Z} \times \mathbb{Z}$ generated by the commuting morphisms $T(z, \theta) = (z + 1, \theta)$ and $S(z, \theta) = (z + \tau + \theta\epsilon, \theta + \delta)$ (not to be confused with the base scheme S). Here ϵ, δ are odd elements of Λ while τ is an even element satisfying $\mathrm{Im}\, \tau_{rd} > 0$. This is a super elliptic curve whose underlying space X_{rd} is the elliptic curve having parameter τ_{rd}. The corresponding $(1|0)$ curve X_0 is the quotient of $\mathbb{C}^{1|0}$ by the morphisms $T_0(z) = z + 1$, $S_0(z) = z + \tau$. The dual curve $\hat{X}$ is easily computed as the quotient of $\mathrm{Spec}(\Lambda[u, \rho])$ by $\hat{T}(u, \rho) = (u + 1, \rho)$ and $\hat{S}(u, \rho) = (u + \tau + \epsilon\delta + \rho\delta, \rho + \epsilon)$, so that duality exchanges ϵ with δ and changes τ to $\tau + \epsilon\delta$. X is projected if $\epsilon = 0$, injected if $\delta = 0$, self-dual (a super Riemann surface) if $\epsilon = \delta$, and split if $\epsilon = \delta = 0$. Only the self-dual case was considered in [13]. The superdiagonal Δ is the quotient of $\mathbb{C}^{1|2}$ by $T_\Delta(z, \theta, \rho) = (z + 1, \theta, \rho)$ and $S_\Delta(z, \theta, \rho) = (z + \tau + \theta\epsilon, \theta + \delta, \rho + \epsilon)$.

We begin by determining the relevant cohomology of these curves. In case $\epsilon = \delta = 0$, $H^0(X, \mathcal{O}_X)$ consists of functions $A(z)+\theta\alpha(z)$ where A and α are constants in Λ, since any nonconstant term in $A(z)$ or $\alpha(z)$ of lowest degree in the generators of Λ would give a nonconstant function on X_{rd}, which is impossible. For general ϵ, δ, $H^0(X, \mathcal{O}_X)$ must be a submodule of this [6]. Clearly, these functions are invariant under the generator S iff $\delta\alpha = 0$, so that $H^0(X, \mathcal{O}_X) = \Lambda|\mathrm{ann}(\delta)$. $H^1(X, \mathcal{O}_X)$ is determined by Serre duality, since the dualizing Berezinian sheaf of X is trivial, but a direct computation via group cohomology will provide more information, so we sketch it here [13].

$H^1(X, \mathcal{O}_X) \cong H^1(G, \mathcal{O}) \cong H^1((S), \mathcal{O}^T)$, where (S) is the cyclic subgroup generated by S, $\mathcal{O}$ are the functions on $\mathbb{C}^{1|1}$, and $\mathcal{O}^T$ are the T-invariant functions. Geometrically this says that the cohomology of X can be computed from the trivial cohomology of the cylinder arising from the quotient by (T), by identifying its ends with S. A cocycle in $H^1((S), \mathcal{O}^T)$ assigns to the generator S a T-invariant function $A(z) + \theta\alpha(z)$, which is regarded as trivial if it can be written as $F(z, \theta) - F(S(z, \theta))$ for some T-invariant function $F(z, \theta) = f(z) + \theta\phi(z)$. This triviality condition implies

$$A(z) = f(z) - f(z + \tau) - \delta\phi(z + \tau),$$
$$\alpha(z) = \phi(z) - \phi(z + \tau) - \epsilon f'(z + \tau) - \epsilon\delta\phi'(z + \tau). \tag{10.1}$$

Since all functions appearing are T-invariant, they have Fourier expansions of the form $A(z) = \sum_n A_n \exp 2\pi i n z$, etc. Rewriting (10.1) in terms of the Fourier components A_n, α_n shows that only the constant modes A_0, α_0 can be nontrivial, in agreement with the expectation from Serre duality. For constant functions, (10.1) immediately reduces to $A = -\delta\phi$. Thus we have $H^1(X, \mathcal{O}_X) = (\Lambda/\delta\Lambda)|\Lambda$. The cohomology of the dual curve $\hat{X}$ has the same form with ϵ replacing δ, namely $H^1(\hat{X}, \mathcal{O}_{\hat{X}}) = (\Lambda/\epsilon\Lambda)|\Lambda$.

With the usual exponential sheaf sequence, implying

$$\mathrm{Pic}^0(X) = H^1(X, (\mathcal{O}_X)_0)/H^1(X, \mathbb{Z})$$

this has the following interpretation. Line bundles of degree zero on X can be specified by multipliers which are trivial for the cycle T and of the form $\exp(A+\theta\alpha)$ for the cycle S, with A and α even and odd elements of Λ respectively. Such a bundle is trivial when $\alpha = 0$ and A is a multiple of δ. For the dual curve, bundles having $\alpha = 0$ and A a multiple of ϵ are trivial. Recall that Proposition 9.8 says that $\hat{\pi}_+\pi^*$ relates bundles having the same constant transition functions on X and $\hat{X}$. This gives an example of a class in $H^1(X, \Lambda^*)$ defining the trivial bundle on X and a nontrivial bundle on $\hat{X}$. The existence of such classes was pointed out in [14].

Our computation also allows us to determine which bundles in $\mathrm{Pic}^0(X)$ admit flat connections, by finding the image of $H^1(X, \Lambda)$ in $H^1(X, \mathcal{O}_X)$. A cocycle for $H^1(G, \Lambda)$ assigns elements of Λ to the generators of G, say $T \mapsto -n$, $S \mapsto m$. (The notation reflects the fact that this computation also determines the image of $H^1(X, \mathbb{Z})$.) To compare with our presentation of $H^1(G, \mathcal{O})$, we subtract a trivial cocycle to set $n = 0$, namely $F(z, \theta) = nz$. The result is $S \mapsto (m + n\tau) + \theta n\epsilon$. That is, the bundles on X admitting flat connections have S-multipliers $\exp(A + \theta\alpha)$ with α a multiple of ϵ.

We can similarly compute the cohomology of Δ in this example. Global functions on Δ have the form $A+\theta\alpha+\rho\beta+\theta\rho B$ with $A, B, \alpha, \beta \in \Lambda$. We find that $H^0(\Delta, \mathcal{O}_\Delta)$ is the submodule of $\Lambda^2|\Lambda^2$ given by the conditions $\delta B = \epsilon B = 0$, $\delta\alpha + \epsilon\beta = 0$. Cocycles for $H^1(\Delta, \mathcal{O}_\Delta)$ have the same form, with the trivial ones generated by $\delta\Lambda \cup \epsilon\Lambda$ and multiples of $\theta\epsilon - \rho\delta$. Thus, for example, bundles on X having multiplier $\exp A$ with $A \in \epsilon\Lambda$ would be trivial on $\hat{X}$ and lift to trivial bundles on Δ; in addition bundles on X having multiplier $\exp\theta\epsilon$ and bundles on $\hat{X}$ having multiplier $\exp\rho\delta$ lift to the same bundle on Δ.

To illustrate Theorem 9.5 we determine the closed one-forms on X; these have the form $\omega = \mathrm{d}zA + \mathrm{d}\theta B$ where $A, B \in \Lambda$ and G-invariance requires $A\epsilon = 0$. (The form $\mathrm{d}z\delta + \mathrm{d}\theta\theta\epsilon$ is also global, but not closed.) Observe that $H^0(X, \Omega^1_{X,cl}) \simeq H^0(\hat{X}, Ber_{\mathcal{O}_{\hat{X}}}) \simeq H^0(\hat{X}, \mathcal{O}_{\hat{X}})$ as required by Corollary 9.4. Working through the proof to compute $\widehat{\mathcal{O}^{\omega}_X}$ we have $\phi_0 = 1 - \theta B - \theta\rho A$. Then the multiplier for $\widehat{\mathcal{O}^{\omega}_X}$ is $(\phi_0 \circ S)/\phi_0$, namely $\exp -\delta(B + \rho A)$, which does indeed belong to the image of $H^1(\hat{X}, \Lambda)$. In the case $\delta = 0$ when X is injected, the transform of the trivial bundle is still trivial, but not in general.

To illustrate the other theorems, specialize to the case of X projected, $\epsilon = 0$. Then Theorem 9.7 describes the transform of the pullback to X of the trivial bundle with connection on X_0. Since the closed one-forms on X_0 have the form $\omega = \mathrm{d}zA$, this is the special case $B = 0$ of the result just obtained: the transform has multiplier $\exp -\delta\rho A$.

For Theorem 9.6, begin with $\mathcal{O}^{\omega}_X$ where $\omega = \mathrm{d}zA + \mathrm{d}\theta B$ and there is no restriction on A, B in this projected situation. The image of ω in $\Omega^1_{X/X_0,cl}$ is $\mathrm{d}\theta\, B$ and we have $\phi_0 = 1 - \theta B$. From the change in ϕ_0 under S we find the multiplier $\exp -\delta B$ for the direct image bundle on X_0.

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