SUPER CURVES, THEIR JACOBIANS, AND SUPER KP EQUATIONS

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Abstract. We study the geometry and cohomology of algebraic super curves, using a new contour integral for holomorphic differentials. For a class of super curves ("generic SKP curves") we define a period matrix. We show that the odd part of the period matrix controls the cohomology of the dual curve. The Jacobian of a generic SKP curve is a smooth supermanifold; it is principally polarized, hence projective, if the even part of the period matrix is symmetric. In general symmetry is not guaranteed by the Riemann bilinear equations for our contour integration, so it remains open whether Jacobians are always projective or carry theta functions.

These results on generic SKP curves are applied to the study of algebro-geometric solutions of the super KP hierarchy. The tau function is shown to be, essentially, a meromorphic section of a line bundle with trivial Chern class on the Jacobian, rationally expressible in terms of super theta functions when these exist. Also we relate the tau function and the Baker function for this hierarchy, using a generalization of Cramer's rule to the supercase.

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Date: May 23, 2006.

1. Introduction.

In this paper we study algebraic super curves with a view towards applications to super Kadomtsev-Petviashvili hierarchies (SKP). We deal from the start with super curves X over a nonreduced ground ring Λ, i.e., our curves carry global nilpotent constants. This has as an advantage, compared to super curves over the complex numbers $\mathbb{C}$, that our curves can be nonsplit, but this comes at the price of some technical complications. The main problem is that the cohomology groups of coherent sheaves on our curves should be thought of as finitely generated modules over the ground ring Λ, instead of vector spaces over $\mathbb{C}$. In general these modules are of course not free. Still we have in this situation Serre duality, as explained in Appendix A, the dualizing sheaf being the (relative) berezinian sheaf $\mathcal{B}\mathrm{er}_X$. In applications to SKP there occurs a natural class of super curves that we call generic

SKP curves. For these curves the most important sheaves, the structure sheaf and the dualizing sheaf, have free cohomology. In the later part of the paper we concentrate on these curves.

Super curves exhibit a remarkable duality uncovered in [DRS90]. The projectivized cotangent bundle of any $N = 1$ super curve has the structure of an $N = 2$ super Riemann surface (SRS), and super curves come in dual pairs $X, \hat{X}$ whose associated $N = 2$ SRSs coincide. Further, the (Λ-valued) points of a super curve can be identified with the effective divisors of degree 1 on its dual. Ordinary $N = 1$ SRSs, widely studied in the context of super string theory, are self dual under this duality. By the resulting identification of points with divisors they enjoy many of the properties that distinguish Riemann surfaces from higher-dimensional varieties. By exploiting the duality we extend this good behaviour to all super curves.

In particular we define for all super curves a contour integration for sections of $\mathcal{B}\mathrm{er}_X$, the holomorphic differentials in this situation. The endpoints of a super contour turn out to be not Λ-points of our super curve, but rather irreducible divisors, i.e., Λ-points on the dual curve! For SRSs these notions are the same and our integration is a generalization of the procedure already known for SRSs. We use this to prove Riemann bilinear relations, connecting in this situation periods of holomorphic differentials on our curve X with those on its dual curve.

In case the cohomology of the structure sheaf is free, e.g. if X is a generic SKP curve, we can define a period matrix and use this to define the Jacobian of X as the quotient of a super vector space by a lattice generated by the period matrix. In this case the Jacobian is a smooth supermanifold. A key question is whether the Jacobian of a generic SKP curve admits ample line bundles (and hence embeddings in projective super space), whose sections could serve as the super analog of theta functions. We show that the symmetry of the even part of the period matrix (together with the automatic positivity of the imaginary part of the reduced matrix) is sufficient for this, and construct the super theta functions in this case. We derive some geometric necessary and sufficient conditions for this symmetry to hold, but it is not an automatic consequence of the Riemann bilinear period relations in this super context. Neither do we know an explicit example in which the symmetry fails. The usual proof that symmetry of the period matrix is necessary for existence of a (principal) polarization also fails because crucial aspects of Hodge theory, particularly the Hodge decomposition of cohomology, do not hold for supertori.

The motivation for writing this paper was our wish to generalize the theory of the algebro-geometric solutions to the KP hierarchy of nonlinear PDEs, as described in [SW85] and references therein, to the closest supersymmetric analog, the "Jacobian" super KP hierarchy of Mulase and Rabin [Mul90, Rab91].

In the super KP case the geometric data leading to a solution include a super curve X and a line bundle $\mathcal{L}$ with vanishing cohomology groups over X. For such a line bundle to exist the super curve X must have a structure sheaf $\mathcal{O}_X$ such that the associated split sheaf $\mathcal{O}_X^{\text{split}}$, obtained by putting the global nilpotent constants in Λ equal to zero, is a direct sum $\mathcal{O}_X^{\text{split}} = \mathcal{O}_X^{\text{red}} \oplus \mathcal{N}$, where $\mathcal{O}_X^{\text{red}}$ is the structure sheaf of the underlying classical curve X^{red} and $\mathcal{N}$ is an invertible $\mathcal{O}_X^{\text{red}}$-sheaf of degree zero. We call such an X an SKP curve, and if moreover $\mathcal{N}$ is not isomorphic to $\mathcal{O}_X^{\text{red}}$ we call X a generic SKP curve.

The Jacobian SKP hierarchy describes linear flows $\mathcal{L}(t_i)$ on the Jacobian of X (with even and odd flow parameters). The other known SKP hierarchies, of Manin–Radul [MR85] and Kac–van de Leur [KvdL87] , describe flows on the universal Jacobian over the moduli space of super curves, in which X as well as $\mathcal{L}$ vary with the t_i [Rab91]. These are outside the scope of this paper, although we hope to return to them elsewhere. As in the non-super case, the basic objects in the theory are the (even and odd) Baker functions, which are sections of $\mathcal{L}(t_i)$ holomorphic except for a single simple pole, and a tau function which is a section of the super determinant (Berezinian) bundle over a super Grassmannian $\mathcal{S}$gr. In contrast to the non-super case, we show that the Berezinian bundle has trivial Chern class, reflecting the fact that the Berezinian is a ratio of ordinary determinants. The super tau function descends, essentially, to $\text{Jac}(X)$ as a section of a bundle with trivial Chern class also, and can be rationally expressed in terms of super theta functions when these exist (its reduced part is a ratio of ordinary tau functions). We also obtain a formula for the even and odd Baker functions in terms of the tau function, confirming that one must know the tau function for the more general Kac–van de Leur flows to compute the Baker functions for even the Jacobian flows in this way, cf. [DS90, Tak95]. For this we need a slight extension of Cramer's rule for solving linear equations in even and odd variables, which is developed in an Appendix via the theory of quasideterminants. In another Appendix we use the Baker functions found in [Rab95b] for Jacobian flow in the case of super elliptic curves to compute the corresponding tau function.

Among the problems remaining open we mention the following. First, to obtain a sharp criterion for when a super Jacobian admits ample line bundles — perhaps always? Second, the fact that generic SKP curves have free cohomology is a helpful simplification which allows us to represent their period maps by matrices and results in their Jacobians being smooth supermanifolds. However, our results should generalize to arbitrary super curves with more singular Jacobians. Finally, one should study the geometry of the universal Jacobian and extend our analysis to the SKP system of Kac–van de Leur.

2. Super curves and their Jacobians.

2.1. **Super curves.** Fix a Grassmann algebra Λ over $\mathbb{C}$; for instance we could take $\Lambda = \mathbb{C}[\beta_1, \beta_2, \dots, \beta_n]$, the polynomial algebra generated by n odd indeterminates. Let $(\bullet, \Lambda)$ be the super scheme $\operatorname{Spec}\Lambda$, with underlying topological space a single point $\bullet$.

A smooth compact connected complex super curve over Λ of dimension $(1|N)$ is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of super commutative Λ-algebras over X, equipped with a structure morphism $(X, \mathcal{O}_X) \to (\bullet, \Lambda)$, such that

1. $(X, \mathcal{O}_X^{\mathrm{red}})$ is a smooth compact connected complex curve, algebraic or holomorphic, depending on the category one is working in. Here $\mathcal{O}_X^{\mathrm{red}}$ is the reduced sheaf of $\mathbb{C}$-algebras on X obtained by quotienting out the nilpotents in the structure sheaf $\mathcal{O}_X$,
2. For suitable open sets $U_\alpha \subset X$ and suitable linearly independent odd elements θ_α^i of $\mathcal{O}_X(U_\alpha)$ we have
$$\mathcal{O}_X(U_\alpha) = \mathcal{O}_X^{\mathrm{red}} \otimes \Lambda[\theta_\alpha^1, \theta_\alpha^2, \dots, \theta_\alpha^N].$$

The U_α's above are called coordinate neighborhoods of $(X, \mathcal{O}_X)$ and $(z_\alpha, \theta_\alpha^1, \theta_\alpha^2, \dots, \theta_\alpha^N)$ are called local coordinates for $(X, \mathcal{O}_X)$, if z_α (mod nilpotents) is a local coordinate for $(X, \mathcal{O}_X^{\mathrm{red}})$. On overlaps of coordinate neighborhoods $U_\alpha \cap U_\beta$ we have

$$\begin{aligned} z_\beta &= F_{\beta\alpha}(z_\alpha, \theta_\alpha^j), \\ \theta_\beta^i &= \Psi_{\beta\alpha}^i(z_\alpha, \theta_\alpha^j). \end{aligned} \tag{2.1}$$

Here the $F_{\beta\alpha}$ are even functions and $\Psi_{\beta\alpha}^i$ odd ones, holomorphic or algebraic depending on the category we are using.

Example 2.1.1. A special case is formed by the *split* super curves. For $N = 1$ they are given by transition functions

$$\begin{aligned} z_\beta &= f_{\beta\alpha}(z_\alpha), \\ \theta_\beta &= \theta_\alpha B_{\beta\alpha}(z_\alpha), \end{aligned} \tag{2.2}$$

with $f_{\beta\alpha}(z_\alpha), B_{\beta\alpha}(z_\alpha)$ even holomorphic (or algebraic) functions that are independent of the nilpotent constants in Λ. So in this case the $f_{\beta\alpha}$ are the transition functions for $\mathcal{O}_X^{\mathrm{red}}$ and $\mathcal{O}_X = \mathcal{O}_X^{\mathrm{red}} \otimes \Lambda \mid \mathcal{N} \otimes \Lambda$, where $\mathcal{N}$ is the $\mathcal{O}_X^{\mathrm{red}}$-module with transition functions $B_{\beta\alpha}(z_\alpha)$. Here and henceforth we denote by a vertical | a direct sum of free Λ-modules, with on the left an evenly generated summand and on the right an odd one.

To any super curve $(X, \mathcal{O}_X)$ there is canonically associated a split curve $(X, \mathcal{O}_X^{\mathrm{split}})$ over $\mathbb{C}$: just take $\mathcal{O}_X^{\mathrm{split}} = \mathcal{O}_X \otimes_\Lambda \Lambda/\mathfrak{m} = \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X$, with $\mathfrak{m} = \langle \beta_1, \dots, \beta_n \rangle$ the maximal ideal of nilpotents in Λ. There is a functor from the category of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X^{\mathrm{split}}$-modules that associates to a sheaf $\mathcal{F}$ the *associated split sheaf* $\mathcal{F}^{\mathrm{split}} = \mathcal{F}/\mathfrak{m}\mathcal{F}$. □

A Λ-point of a super curve $(X, \mathcal{O}_X)$ is a morphism $\phi : (\bullet, \Lambda) \to (X, \mathcal{O}_X)$ such that the composition with the structural morphism $(X, \mathcal{O}_X) \to (\bullet, \Lambda)$ is the identity (of $(\bullet, \Lambda)$). Locally, in an open set U_α containing $\phi(\bullet)$, a Λ-point is given by specifying the images under the even Λ-homomorphism $\phi^\sharp : \mathcal{O}_X(U_\alpha) \to \Lambda$ of the local coordinates: $p_\alpha = \phi^\sharp(z_\alpha), \pi_\alpha^i = \phi^\sharp(\theta_\alpha^i)$. The local parameters (p_α, π_α^i) of a Λ-point transform precisely as the coordinates do, see (2.1). By quotienting out nilpotents in a Λ-point (p_α, π_α^i) we obtain a complex number p_α^{red}, the coordinate of the reduced point of $(X, \mathcal{O}_X^{\mathrm{red}})$ corresponding to the Λ-point (p_α, π_α^i).

2.2. **Duality and $N = 2$ curves.** Our main interest is the theory of $N = 1$ super curves but as a valuable tool for the study of these curves we make use of $N = 2$ curves as well in this paper. Indeed, as is well known, [DRS90, Sch94], one can associate in a canonical way to an $N = 1$ curve an (untwisted) super conformal $N = 2$ curve, as we will now recall. The introduction of the super conformal $N = 2$ curve clarifies the whole theory of $N = 1$ super curves.

Let from now on $(X, \mathcal{O}_X)$ be an $N = 1$ super curve. Any invertible sheaf $\mathcal{E}$ for $(X, \mathcal{O}_X)$ and any extension of $\mathcal{E}$ by the structure sheaf:

$$0 \to \mathcal{O}_X \to \hat{\mathcal{E}} \to \mathcal{E} \to 0,$$

defines in the obvious way an $N = 2$ super curve $(X, \hat{\mathcal{E}})$. It has local coordinates $(z_\alpha, \theta_\alpha, \rho_\alpha)$, where $(z_\alpha, \theta_\alpha)$ are local coordinates for $(X, \mathcal{O}_X)$. On overlaps we will have

$$\begin{aligned} z_\beta &= F_{\beta\alpha}(z_\alpha, \theta_\alpha), \\ \theta_\beta &= \Psi_{\beta\alpha}(z_\alpha, \theta_\alpha), \\ \rho_\beta &= H_{\beta\alpha}(z_\alpha, \theta_\alpha)\rho_\alpha + \phi_{\beta\alpha}(z_\alpha, \theta_\alpha). \end{aligned} \tag{2.3}$$

(So $H_{\beta\alpha}(z_\alpha, \theta_\alpha)$ is the transition function for the generators of the invertible sheaf $\mathcal{E}$.) We want to choose the extension (2.2) such that

$(X, \hat{\mathcal{E}})$ is *super conformal*, in the sense that the local differential form $\omega_\alpha = dz_\alpha - d\theta_\alpha \rho_\alpha$ is globally defined up to a scale factor. Now

$$\omega_\beta = dz_\beta - d\theta_\beta \rho_\beta = dz_\alpha(\frac{\partial F}{\partial z_\alpha} - \frac{\partial \Psi}{\partial z_\alpha}\rho_\beta) - d\theta_\alpha(-\frac{\partial F}{\partial \theta_\alpha} + \frac{\partial \Psi}{\partial \theta_\alpha}\rho_\beta).$$

(Here we suppress the subscripts on F and Ψ, as we will do below.) We see that for $\hat{\mathcal{E}}$ to be super conformal we need

$$\rho_\alpha = \frac{(-\frac{\partial F}{\partial \theta_\alpha} + \frac{\partial \Psi}{\partial \theta_\alpha}\rho_\beta)}{(\frac{\partial F}{\partial z_\alpha} - \frac{\partial \Psi}{\partial z_\alpha}\rho_\beta)},$$

or

$$\rho_\beta = \frac{(\frac{\partial F}{\partial \theta_\alpha} + \frac{\partial F}{\partial z_\alpha}\rho_\alpha)}{(\frac{\partial \Psi}{\partial \theta_\alpha} - \frac{\partial \Psi}{\partial z_\alpha}\rho_\alpha)}. \tag{2.4}$$

Conversely one checks that if (2.4) holds for all overlaps the cocycle condition is satisfied and that we obtain in this manner an $N = 2$ super curve. To show that this super curve is an extension as in (2.2), it is useful to note that (2.4) can also be written as

$$\rho_\beta = \operatorname{ber}\begin{pmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{pmatrix} \rho_\alpha + \frac{\partial_\theta F}{\partial_\theta \Psi}. \tag{2.5}$$

The homomorphism ber is defined in Appendix C, see (C.3). Recall that the local generators f_α of the dualizing sheaf (see Appendix A) $\mathcal{B}\mathrm{er}_X$ of $(X, \mathcal{O}_X)$ transform as

$$f_\beta = \operatorname{ber}\begin{pmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{pmatrix} f_\alpha. \tag{2.6}$$

If we denote by $\mathcal{C}\mathcal{O}_X$ the structure sheaf of the super conformal $N = 2$ super curve just constructed, we see that we have an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{C}\mathcal{O}_X \to \mathcal{B}\mathrm{er}_X \to 0. \tag{2.7}$$

$\mathcal{C}\mathcal{O}_X$ is the only extension of $\mathcal{B}\mathrm{er}_X$ by the structure sheaf that is super conformal. This sequence is *trivial* if it isomorphic ([HS71]) to a split sequence.

Definition 2.2.1. A super curve is called *projected* if there is a cover of X such that the transition functions $F_{\beta\alpha}$ in (2.1) are independent of the odd coordinates θ^j_α.

For projected curves we have a projection morphism $(X, \mathcal{O}_X) \to (X, \mathcal{O}_X^{\mathrm{red}} \otimes \Lambda)$ corresponding to the sheaf inclusion $\mathcal{O}_X^{\mathrm{red}} \otimes \Lambda \to \mathcal{O}_X$. This inclusion can be defined only for projected curves.

A projected super curve has a $\mathcal{CO}_X$ that is a trivial extension but the converse is not true, as we will see when we discuss super Riemann surfaces in subsection 2.3. The relation between projectedness of $(X, \mathcal{O}_X)$ and the triviality of the extension defining $(X, \mathcal{CO}_X)$ is discussed in detail in subsection 2.11.

Example 2.2.2. If $(X, \mathcal{O}_X)$ is split, (2.6) becomes

$$f_\beta = \frac{\partial_z f_{\beta\alpha}}{B_{\beta\alpha}} f_\alpha. \tag{2.8}$$

This means that in this case $\mathcal{B}\mathrm{er}_X = \mathcal{K}\mathcal{N}^{-1} \otimes \mathcal{O}_X = \mathcal{K}\mathcal{N}^{-1} \otimes \Lambda \mid \mathcal{K} \otimes \Lambda$, where $\mathcal{K}$ is the canonical sheaf for $\mathcal{O}_X^{\mathrm{red}}$.

Split curves are projected and the sequence (2.7) becomes trivial. As an $\mathcal{O}_X^{\mathrm{red}}$-module we have $\mathcal{CO}_X = (\mathcal{O}_X^{\mathrm{red}} \oplus \mathcal{K}) \otimes \Lambda \mid (\mathcal{N} \oplus \mathcal{K}\mathcal{N}^{-1}) \otimes \Lambda$. □

The map $\mathcal{CO}_X \to \mathcal{B}\mathrm{er}_X$ is locally described by the differential operator $D_{\mathcal{C}}^\alpha = \partial_{\rho_\alpha}$. Indeed, the operator $D_{\mathcal{C}}^\alpha$ transforms homogeneously, $D_{\mathcal{C}}^\beta = \mathrm{ber} \begin{pmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{pmatrix}^{-1} D_{\mathcal{C}}^\alpha$, so this defines a global $(0 \mid 1)$ dimensional distribution $D_{\mathcal{C}}$ and the quotient of $(X, \mathcal{CO}_X)$ by this distribution is precisely $(X, \mathcal{O}_X)$.

Now the distribution $D_{\mathcal{C}}$ annihilates the 1-form ω used to find $\mathcal{CO}_X$. This form locally looks like $\omega_\alpha = dz_\alpha - d\theta_\alpha \rho_\alpha$ and its kernel is generated by $D_{\mathcal{C}}^\alpha$ and a second operator $\hat{D}_{\mathcal{C}}^\alpha = \partial_{\theta_\alpha} + \rho_\alpha \partial_{z_\alpha}$. (The operators that we call $D_{\mathcal{C}}$ and $\hat{D}_{\mathcal{C}}$ are in the literature also denoted by D^+ and D^-, cf. [DRS90]) To study the result of "quotienting by the distribution $\hat{D}_{\mathcal{C}}$" we introduce in each coordinate neighborhood U_α new coordinates:

$$\begin{aligned} \hat{z}_\alpha &= z_\alpha - \theta_\alpha \rho_\alpha, \\ \hat{\theta}_\alpha &= \theta_\alpha, \\ \hat{\rho}_\alpha &= \rho_\alpha. \end{aligned}$$

In the sequel we will drop the hats ˆ on θ and ρ, hopefully not causing too much confusion.

In these new coordinates we have

$$\hat{D}_{\mathcal{C}}^\alpha = \partial_{\theta_\alpha}, \quad D_{\mathcal{C}}^\alpha = \partial_{\rho_\alpha} + \theta_\alpha \partial_{\hat{z}_\alpha}.$$

So the kernel of $\hat{D}_{\mathcal{C}}$ consists locally of functions of $\hat{z}_\alpha, \rho_\alpha$. To see that this makes global sense we observe that

$$\begin{aligned} \hat{z}_\beta &= F(\hat{z}_\alpha, \rho_\alpha) + \frac{DF(\hat{z}_\alpha, \rho_\alpha)}{D\Psi(\hat{z}_\alpha, \rho_\alpha)} \Psi(\hat{z}_\alpha, \rho_\alpha), \\ \rho_\beta &= \frac{DF(\hat{z}_\alpha, \rho_\alpha)}{D\Psi(\hat{z}_\alpha, \rho_\alpha)}, \end{aligned} \tag{2.9}$$

where $D = \partial_\theta + \theta\partial_z$. The details of this somewhat unpleasant calculation are left to the reader. From (2.9) we see that $\mathcal{CO}_X$ contains the structure sheaf $\hat{\mathcal{O}}_X$ of another $N = 1$ super curve: $\hat{\mathcal{O}}_X$ is the sheaf of Λ-algebras locally generated by $\hat{z}_\alpha, \rho_\alpha$. We call $\hat{X} = (X, \hat{\mathcal{O}}_X)$ the *dual curve* of $(X, \mathcal{O}_X)$. We have

$$0 \to \hat{\mathcal{O}}_X \to \mathcal{CO}_X \xrightarrow{\hat{D}_{\mathcal{C}}} \mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X \to 0, \tag{2.10}$$

where $\mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X$ is the dualizing sheaf of the dual curve. One can show that the dual curve of the dual curve is the original curve, thereby justifying the terminology.

Example 2.2.3. We continue the discussion of split curves. In this case (2.9) becomes

$$\begin{aligned} \hat{z}_\beta &= f(\hat{z}_\alpha), \\ \rho_\beta &= \frac{\partial_{\hat{z}} f(\hat{z}_\alpha)}{B(\hat{z}_\alpha)} \rho_\alpha, \end{aligned} \tag{2.11}$$

So the dual split curve is $\hat{\mathcal{O}}_X^{\mathrm{split}} = \mathcal{O}_X^{\mathrm{red}} \otimes \Lambda \mid \mathcal{K}\mathcal{N}^{-1} \otimes \Lambda$. The Berezinian sheaf for the dual split curve has generators that satisfy

$$\hat{f}_\beta = B(z_\alpha) \hat{f}_\alpha. \tag{2.12}$$

This means that $\mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X = \mathcal{N} \otimes \hat{\mathcal{O}}_X = \mathcal{N} \otimes \Lambda \mid \mathcal{K} \otimes \Lambda$. □

A very useful geometric interpretation of the dual curve exists, cf. [DRS90, Sch94]: the points (i.e., the Λ-points) of the dual curve correspond precisely to the irreducible divisors of the original curve and vice versa, as we will presently discuss. In subsection 2.4 we will see that irreducible divisors are the limits that occur in contour integration on a super curve.

An irreducible divisor (for $\mathcal{O}_X$) is locally given by an even function $P_\alpha = z_\alpha - \hat{z}_\alpha - \theta_\alpha \rho_\alpha \in \mathcal{O}_X(U_\alpha)$, where $\hat{z}_\alpha$ and ρ_α are now respectively even and odd constants, i.e., elements of Λ. Two divisors P_α, P_β defined on coordinate neighborhoods U_α and U_β, respectively, are said to correspond to each other on the overlap if

$$P_\beta(z_\beta, \theta_\beta) = P_\alpha(z_\alpha, \theta_\alpha) g(z_\alpha, \theta_\alpha), \quad g(z_\alpha, \theta_\alpha) \in \mathcal{O}_{X,\mathrm{ev}}^\times(U_\alpha \cap U_\beta). \tag{2.13}$$

(If R is a ring (or sheaf of rings) $R^\times$ is the set of invertible elements.)

Lemma 2.2.4. *Let $(U, \mathcal{O}(U))$ be a $(1 \mid 1)$ dimensional super domain with coordinates (z, θ) and let $f(z, \theta) \in \mathcal{O}(U)$. Then, with $D = \partial_\theta + \theta\partial_z$,*

$$f(z, \theta) = (z - \hat{z} - \theta\rho) g(z, \theta) \quad \Leftrightarrow \quad f(\hat{z}, \rho) = 0, Df(\hat{z}, \rho) = 0,$$

for $g(z, \theta)$ in $\mathcal{O}(U)$.

Applying Lemma 2.2.4 to (2.13) we find

$$
\begin{aligned}
P_\beta(F(\hat z_\alpha,\rho_\alpha),\Psi(\hat z_\alpha,\rho_\alpha)) &= F(\hat z_\alpha,\rho_\alpha)-\hat z_\beta-\Psi(\hat z_\alpha,\rho_\alpha)\rho_\beta=0,\\
DP_\beta(F(\hat z_\alpha,\rho_\alpha),\Psi(\hat z_\alpha,\rho_\alpha)) &= DF(\hat z_\alpha,\rho_\alpha)-D\Psi(\hat z_\alpha,\rho_\alpha)\rho_\beta=0.
\end{aligned}
$$

From this one sees that the parameters $(\hat z_\alpha,\rho_\alpha)$ in the local expression for an irreducible divisor transform as in (2.9), so they are Λ-points of the dual curve.

The $N=2$ super conformal super curve canonically associated to a super curve has a structure sheaf $\mathcal{CO}_X$ that comes equipped with two sheaf maps $D_{\mathcal{C}}$ and $\hat D_{\mathcal{C}}$ with kernels the structure sheaves $\mathcal{O}_X$ and $\hat{\mathcal{O}}_X$ of the original super curve and its dual. The intersection of the kernels is the constant sheaf Λ. The images of these maps are the dualizing sheaves $\mathcal{B}\mathrm{er}_X$ and $\mathcal{B}\hat{\mathrm{er}}_X$. In fact we can restrict $D_{\mathcal{C}},\hat D_{\mathcal{C}}$ to the subsheaves $\hat{\mathcal{O}}_X$ and $\mathcal{O}_X$, respectively, without changing the images. This gives us exact sequences

$$
\begin{aligned}
&0\to\Lambda\to\mathcal{O}_X\xrightarrow{\hat D}\mathcal{B}\hat{\mathrm{er}}_X\to 0,\\
&0\to\Lambda\to\hat{\mathcal{O}}_X\xrightarrow{D}\mathcal{B}\mathrm{er}_X\to 0,
\end{aligned}
\tag{2.14}
$$

with $D=D_{\mathcal{C}}|_{\hat{\mathcal{O}}_X}$ and $\hat D=\hat D_{\mathcal{C}}|_{\mathcal{O}_X}$. Just as the sheaf maps $D_{\mathcal{C}},\hat D_{\mathcal{C}}$ have local expressions as differential operators, also their restrictions are locally expressible in terms of differential operators: if $\{f_\alpha(z_\alpha,\theta_\alpha)\}$ is a section of $\mathcal{O}_X$ then the corresponding section $\{(\hat D_{\mathcal{C}}f_\alpha)(\hat z_\alpha,\rho_\alpha)\}$ of $\mathcal{B}\hat{\mathrm{er}}_X$ is given by

$$
\hat D f_\alpha(\hat z_\alpha,\rho_\alpha)=[(\partial_\theta+\theta\partial_z)f_\alpha]|_{z_\alpha=\hat z_\alpha,\theta_\alpha=\rho_\alpha}.
$$

Similarly, if $\{\hat f_\alpha(\hat z_\alpha,\rho_\alpha)\}$ is a section of $\hat{\mathcal{O}}_X$ then the corresponding section of $\mathcal{B}\mathrm{er}_X$ is

$$
D\hat f_\alpha(z_\alpha,\theta_\alpha)=[(\partial_\rho+\rho\partial_{\hat z})\hat f_\alpha]|_{\hat z_\alpha=z_\alpha,\rho_\alpha=\theta_\alpha}.
$$

We summarize the relationships between the various sheaves and sheaf maps in the following commutative diagram (of sheaves of Λ-algebras):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & \hat{\mathcal{O}}_X & \xrightarrow{D} & \mathcal{B}\mathrm{er}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{C}\mathcal{O}_X & \xrightarrow{D_{\mathcal{C}}} & \mathcal{B}\mathrm{er}_X & \longrightarrow & 0 \\
 & & {\scriptstyle \hat{D}}\downarrow & & {\scriptstyle \hat{D}_{\mathcal{C}}}\downarrow & & & & \\
 & & \mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X & = & \mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{2.15}
$$

We conclude this subsection with the remark that the dualizing sheaf

$\mathcal{B}\mathrm{er}(\mathcal{C}\mathcal{O}_X)$ of the super conformal super curve $(X, \mathcal{C}\mathcal{O}_X)$ associated to a super curve $(X, \mathcal{O}_X)$ is trivial, making $(X, \mathcal{C}\mathcal{O}_X)$ a super analog of an elliptic curve or a Calabi-Yau manifold, cf. [DN91]. In fact, this statement is true for any $N = 2$ super curve $(X, \mathcal{E})$ where $\mathcal{E}$ is an extension of $\mathcal{B}\mathrm{er}_X$ by the structure sheaf: if we have

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{B}\mathrm{er}_X \to 0,$$

then $\mathcal{E}$ has local generators $(z_\alpha, \theta_\alpha, \rho_\alpha)$ on U_α, and on overlaps we get

$$
\begin{aligned}
z_\beta &= F_{\beta\alpha}(z_\alpha, \theta_\alpha), \\
\theta_\beta &= \Psi_{\beta\alpha}(z_\alpha, \theta_\alpha), \\
\rho_\beta &= \Phi_{\beta\alpha}(z_\alpha, \theta_\alpha, \rho_\alpha) = \mathrm{ber}(J(z, \theta))\rho_\alpha + \phi_{\beta\alpha}(z_\alpha, \theta_\alpha),
\end{aligned}
\tag{2.16}
$$

where $\mathrm{ber}(J(z, \theta))$ is the Berezinian of the super Jacobian matrix of the change of (z, θ) coordinates; this is precisely the transition function for $\mathcal{B}\mathrm{er}_X$, see (2.6). Then the super Jacobian matrix

$$
J(z, \theta, \rho) = \mathrm{ber} \begin{pmatrix} \partial_z F & \partial_z \Psi & \partial_z \Phi \\ \partial_\theta F & \partial_\theta \Psi & \partial_\theta \Phi \\ \partial_\rho F & \partial_\rho \Psi & \partial_\rho \Phi \end{pmatrix} = \mathrm{ber} \begin{pmatrix} \partial_z F & \partial_z \Psi & \partial_z \Phi \\ \partial_\theta F & \partial_\theta \Psi & \partial_\theta \Phi \\ 0 & 0 & \partial_\rho \Phi \end{pmatrix} =
$$
$$
= \mathrm{ber}(J(z, \theta))/\partial_\rho \Phi = 1,
$$

for all overlaps $U_\alpha \cap U_\beta$, and therefore $(X, \mathcal{E})$ has trivial dualizing sheaf.

2.3. **Super Riemann surfaces.** In this subsection we briefly discuss a special class of $N = 1$ super curves, the super Riemann surfaces (SRS). This class of curves is studied widely in the literature because of its applications in super string theory, see e.g., [Fri86, GN88a, LR88, CR88]. (Also the term SUSY_1 curve is used, [Man88, Man91], or super conformal manifold, [RSV88].) From our point of view super Riemann surfaces are special because irreducible divisors and Λ-points can be identified and because there is a differential operator taking functions to sections of the dualizing sheaf. Both facts simplify the theory considerably. However, by systematically using the duality of the $N = 2$ super conformal curve one can extend results previously obtained solely for super Riemann surfaces to arbitrary super curves.

In the previous subsection we have seen that every $N = 1$ super curve $(X, \mathcal{O}_X)$ has a dual curve $(X, \hat{\mathcal{O}}_X)$. Of course it can happen that the transition functions of $(X, \mathcal{O}_X)$ are identical to those of the dual curve $(X, \hat{\mathcal{O}}_X)$. This occurs if the transition functions satisfy

$$DF(z_\alpha, \theta_\alpha) = \Psi(z_\alpha, \theta_\alpha) D\Psi(z_\alpha, \theta_\alpha). \tag{2.17}$$

If (2.17) holds then the operator $D_\alpha = \partial_{\theta_\alpha} + \theta_\alpha \partial_{z_\alpha}$ transforms as

$$D_\beta = (D\Psi)^{-1} D_\alpha \tag{2.18}$$

So in the situation of (2.17) the super curve $(X, \mathcal{O}_X)$ carries a $(0 \mid 1)$ dimensional distribution D such that D^2 is nowhere vanishing (in fact $D^2 = \partial_z$). A super curve carrying such a distribution is called a $(N = 1)$ super Riemann surface. Equivalently an $N = 1$ super Riemann surface is a $(N = 1)$ super curve that carries an odd global differential operator with nowhere vanishing square that takes values in some invertible sheaf.

Recall the Berezinian that occurs in the transformation law for generators of $\mathcal{B}\mathrm{er}_X$, (2.6). It can be written in general as

$$\operatorname{ber} \begin{pmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{pmatrix} = D(\frac{DF}{D\Psi}).$$

Therefore if (2.17) holds we have $\operatorname{ber} \begin{pmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{pmatrix} = D\Psi$ so (2.18) tells us that D takes values in the dualizing sheaf $\mathcal{B}\mathrm{er}_X$.

So super Riemann surfaces are self dual, as probably first noted in [DRS90]. More generally, the question then arises what happens if the curves $(X, \mathcal{O}_X)$ and $(X, \hat{\mathcal{O}}_X)$ are isomorphic, but a priori not with identical transition functions. We claim that also in this case the curve $(X, \mathcal{O}_X)$ is a super Riemann surface. Indeed, the operator $\hat{D}_{\mathcal{C}}$ restricted to $\mathcal{O}_X$ takes values in the dualizing sheaf $\mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X$ of $\hat{\mathcal{O}}_X$, as we have seen above. Using the isomorphism we can think of $\hat{D}_{\mathcal{C}}$ as a differential operator taking values in a sheaf isomorphic to the dualizing sheaf $\mathcal{B}\mathrm{er}_X$ on $\mathcal{O}_X$. Since $\hat{D}_{\mathcal{C}}^2$ does not vanish we see that $(X, \mathcal{O}_X)$ is a super Riemann surface. Now it is known (and easy to see) that for any super Riemann surface there are coordinates such that (2.17) holds. In these coordinates the transition functions of $(X, \mathcal{O}_X)$ and $(X, \hat{\mathcal{O}}_X)$ are in fact equal.

The $N = 2$ super conformal curve $(X, \mathcal{C}\mathcal{O}_X)$ associated to a SRS $(X, \mathcal{O}_X)$ is very simple. Recall that $\mathcal{C}\mathcal{O}_X$ is an extension

$$0 \to \mathcal{O}_X \to \mathcal{C}\mathcal{O}_X \xrightarrow{\epsilon} \mathcal{B}\mathrm{er}_X \to 0.$$

where locally $\epsilon(z) = \epsilon(\theta) = 0$ and $\epsilon(\rho) = f$, with f a local generator of $\mathcal{B}\mathrm{er}_X$. For SRS there is a splitting $e : \mathcal{B}\mathrm{er}_X \to \mathcal{C}\mathcal{O}_X$, given locally by $e(f) = \rho - \theta$. One needs to use the definition of a SRS to check that this definition makes global sense, i.e., that $\rho - \theta$ transforms as a section of $\mathcal{B}\mathrm{er}_X$; for this see [Rab95a]. In other words for a SRS the associated $N = 2$ curve has a split structure sheaf:

$$\mathcal{C}\mathcal{O}_X = \mathcal{O}_X \oplus \mathcal{B}\mathrm{er}_X.$$

Note that not all SRS's are projected, so there are examples where $\mathcal{C}\mathcal{O}_X$ is a trivial extension but where $(X, \mathcal{O}_X)$ is not projected.

2.4. **Integration on super curves.** Let us first recall the classical situation. On an ordinary Riemann surface $(X, \mathcal{O}_X^{\mathrm{red}})$ we can integrate a holomorphic 1-form ω along a contour connecting two points p and q on X. If the contour connecting p and q lies in a single, simply connected, coordinate neighborhood U_α with local coordinate z_α we can write $\omega = df_\alpha$, with $f_\alpha \in \mathcal{O}_X^{\mathrm{red}}(U_\alpha)$ determined up to a constant. The points p, q are described by the irreducible divisors $z_\alpha - p_\alpha$ and $z_\alpha - q_\alpha$. Then we calculate the integral of ω along the contour by $\int_p^q \omega = f_\alpha(q_\alpha) - f_\alpha(p_\alpha)$. Suppose next that p and q are in different coordinate neighborhoods U_α and U_β, with coordinates z_α, z_β related by $z_\beta = F(z_\alpha)$ on overlaps. Assume furthermore that the contour connecting them contains a point $r \in U_\alpha \cap U_\beta$. Then we can write $\omega = df_\alpha$ on U_α, and $\omega = df_\beta$ on U_β, with $f_\alpha(z_\alpha) = f_\beta(F(z_\alpha)) + c_{\alpha\beta}$ on overlaps, where $c_{\alpha\beta}$ is locally constant on $U_\alpha \cap U_\beta$. The intermediate point r can be described by two (corresponding) irreducible divisors $z_\alpha - r_\alpha$ and $z_\beta - r_\beta$. Then $\int_p^q \omega = \int_p^r \omega + \int_r^q \omega = f_\beta(q_\beta) - f_\beta(r_\beta) + f_\alpha(r_\alpha) - f_\alpha(p_\alpha)$. This is independent of the intermediate point because the parameter r_α in the irreducible divisor $z_\alpha - r_\alpha$ transforms as a $\mathbb{C}$-point of the curve: we have $r_\beta = F(r_\alpha)$, and $f_\alpha(r_\alpha) - f_\beta(r_\beta) = c_{\alpha\beta}$; therefore we can replace r by any other intermediate point in the same connected component of

$U_\alpha \cap U_\beta$. If p and q are not in adjacent coordinate neighborhoods we need to introduce more intermediate points.

So there are three crucial facts in the construction of the contour integral of holomorphic 1-forms on a Riemann surface: the parameter in an irreducible divisor transforms as a point, d is an operator that produces from a function on X a section of the dualizing sheaf on X, and the kernel of the operator d consists of the constants. We will find analogs of all three facts for super curves.

We have seen that for an $N = 1$ super curve in general the parameters in an irreducible divisor correspond to a Λ-point of the dual curve. Also the sheaf map D acting on the dual curve maps sections of $\hat{\mathcal{O}}_X$ to sections of $\mathcal{B}\mathrm{er}_X$, see (2.15).

This suggests that we define a *(super) contour* $\Gamma = (\gamma, P, Q)$ on $(X, \mathcal{O}_X)$ as an ordinary contour γ on the underlying topological space X, together with two irreducible divisors P and Q for $(X, \mathcal{O}_X)$ such that the reduced divisors of P and Q are the endpoints of γ. So if

$$P = z - \hat{p} - \theta\hat{\pi}, \quad Q = z - \hat{q} - \theta\hat{\chi},$$

then the corresponding Λ-points on the dual curve $(X, \hat{\mathcal{O}}_X)$ are $(\hat{p}, \hat{\pi})$, $(\hat{q}, \hat{\chi})$, and $z = \hat{p}^{\mathrm{red}}$ and $z = \hat{q}^{\mathrm{red}}$ are the equations for the endpoints of the curve γ. Then we define the integral of a section $\{\omega_\alpha = D\hat{f}_\alpha\}$ of the dualizing sheaf on $(X, \mathcal{O}_X)$ along Γ by

$$\int_P^Q \omega = \int_P^Q D\hat{f} = \hat{f}(\hat{q}, \hat{\chi}) - \hat{f}(\hat{p}, \hat{\pi}).$$

Here we assume that the contour connecting P and Q lies in a single simply connected open set. If the contour traverses various open sets we need to choose intermediate divisors on the contour, as before.

A super contour Γ is called *closed* if it is of the form $\Gamma = (\gamma, P, P)$, with the underlying contour γ closed in the usual sense. Observe that the integral over Γ is independent of the choice of P, so we will omit reference to it.

The contour integration on $N = 1$ super curves introduced here seems to be new; it is a nontrivial generalization of the contour integral on super Riemann surfaces, as described for instance in [Fri86, McA88, Rog88].

For closed contours it agrees with the integration theory described in [GKS83, Khu95].

We can also understand this integration procedure in terms of the contour integral on the $N = 2$ super conformal super curve $(X, \mathcal{CO}_X)$, introduced by Cohn, [Coh87]. To this end define on $\mathcal{CO}_X(U) \oplus \mathcal{CO}_X(U)$

the sheaf map $(D_{\mathcal{C}}, \hat{D}_{\mathcal{C}})$ by the local componentwise action of the differential operators $D^{\alpha}_{\mathcal{C}}$ and $\hat{D}^{\alpha}_{\mathcal{C}}$ as before. Then the square of the operator $(D_{\mathcal{C}}, \hat{D}_{\mathcal{C}})$ vanishes and the Poincaré Lemma holds for $(D_{\mathcal{C}}, \hat{D}_{\mathcal{C}})$:

Lemma 2.4.1. *Let U be a simply connected open set on X and let $(f, g) \in \mathcal{CO}_X(U) \oplus \mathcal{CO}_X(U)$ such that $(D_{\mathcal{C}}, \hat{D}_{\mathcal{C}})(f, g) = 0$. Then there is an element $H \in \mathcal{CO}_X(U)$, unique up to an additive constant, such that*

$$(f, g) = (D_{\mathcal{C}} H, \hat{D}_{\mathcal{C}} H).$$

Let then $\mathcal{M}(U) \subset \mathcal{CO}_X(U) \oplus \mathcal{CO}_X(U)$ be the subsheaf of $(D_{\mathcal{C}}, \hat{D}_{\mathcal{C}})$-closed sections. Note that a section of $\mathcal{M}$ looks in U_α like $(f_\alpha, g_\alpha) = (f(z_\alpha, \theta_\alpha), g(\hat{z}_\alpha, \rho_\alpha))$ and furthermore f is a section of $\mathcal{B}\mathrm{er}_X$ and g is a section of $\mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X$. This means that $\mathcal{M}$ globalizes to the direct sum $\mathcal{B}\mathrm{er}_X \oplus \mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X$.

So we get an exact sequence of sheaves:

$$0 \to \Lambda \to \mathcal{CO}_X \overset{(}{D}_{\mathcal{C}}, \hat{D}_{\mathcal{C}}) \to \mathcal{M} \to 0.$$

Now the sections of $\mathcal{M}$ are the objects on $\mathcal{CO}_X$ that can be integrated. A contour for $\mathcal{CO}_X$ is a triple $(\gamma, \mathcal{C}P, \mathcal{C}Q)$ where $\mathcal{C}P, \mathcal{C}Q$ are two Λ-points of $(X, \mathcal{CO}_X)$ with as reduced points the endpoints of the contour γ. Assume that the contour lies in a single simply connected open set U. If $\omega \in \mathcal{M}(U)$ then we can write $\omega = (D_{\mathcal{C}} H, \hat{D}_{\mathcal{C}} H)$ for some $H \in \mathcal{CO}_X(U)$ and we put $\int_{\mathcal{C}P}^{\mathcal{C}Q} \omega = H(\mathcal{C}Q) - H(\mathcal{C}P)$. Extension to more complicated contours as before.

Now start with a section $\{s_\alpha\}$ of $\mathcal{B}\mathrm{er}_X$ on $(X, \mathcal{O}_X)$. We can lift it to the section $\{(s_\alpha, 0)\}$ of $\mathcal{M}$. In particular there is a section $\{H_\alpha\}$ of $\mathcal{CO}_X$ such that $s_\alpha = D_{\mathcal{C}} H_\alpha$, $\hat{D}_{\mathcal{C}} H_\alpha = 0$. This means that $\{H_\alpha\}$ is in fact a section of the subsheaf $\hat{\mathcal{O}}_X$. So in specifying the Λ-points of $(X, \mathcal{CO}_X)$ on the ends of the contour we have the freedom to shift along the fiber of the projection $\hat{\pi} : (X, \mathcal{CO}_X) \to (X, \hat{\mathcal{O}}_X)$. In other words we only need to specify Λ-points of the dual curve, or, equivalently, irreducible divisors on the original curve.

Therefore we can define the integral of a section $s = \{s_\alpha\}$ of $\mathcal{B}\mathrm{er}_X$ along a contour with P, Q two irreducible divisors for $(X, \mathcal{O}_X)$ at the end point as follows. We choose two Λ-points $\mathcal{C}P$ and $\mathcal{C}Q$ of $(X, \mathcal{CO}_X)$ that project to the Λ-points of $(X, \hat{\mathcal{O}}_X)$ corresponding to P, Q. Then $\int_P^Q s = H(\mathcal{C}Q) - H(\mathcal{C}P)$ if $s_\alpha = D_{\mathcal{C}} H$ and $\hat{D}_{\mathcal{C}} H = 0$. Again we are assuming here that the contour lies in a simply connected region and extend for the general case using intermediate points. One checks that this procedure of integrating a section of the dualizing sheaf on $(X, \mathcal{O}_X)$ using integration on $\mathcal{CO}_X$ is the same as we had defined before.

2.5. **Integration on the universal cover.** We consider from now on only holomorphic (compact, connected, $N = 1$) super curves $(X, \mathcal{O}_X)$ of genus $g > 1$. We fix a point $x_0 \in X$ and 1-cycles $A_i, B_i, i = 1, \dots, g$ through x_0 with intersection $A_i \cdot B_j = \delta_{ij}$, $A_i \cdot A_j = B_i \cdot B_j = 0$ as usual. Then the fundamental group $\pi_1(X, x_0)$ is generated by the classes a_i, b_i corresponding to the loops A_i, B_i, subject solely to the relation $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = e$.

The universal cover of the super curve $(X, \mathcal{O}_X)$ is the open superdisk $D^{1|1} = (D, \mathcal{O}_{D^{1|1}})$ of dimension $(1 \mid 1)$, where $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathcal{O}_{D^{1|1}} = \mathcal{O}_D \otimes_{\mathbb{C}} \Lambda[\theta]$, with $\mathcal{O}_D$ the usual sheaf of holomorphic functions on the unit disk. The group G of covering transformations of $(D, \mathcal{O}_{D^{1|1}}) \to (X, \mathcal{O}_X)$ is isomorphic to $\Pi_1(X, x_0)$ and each covering transformation g is determined by its action on the global coordinates (z, θ) of $D^{1|1}$. Introduce super holomorphic functions by

$$F_g(z, \theta) := g^{-1} \cdot z, \quad \Psi_g(z, \theta) := g^{-1} \cdot \theta.$$

If P_p is a Λ-point of $D^{1|1}$, i.e., a homomorphism $\mathcal{O}_{D^{1|1}} \to \Lambda$, determined by $z \mapsto z_P \in \Lambda_0, \theta \mapsto \theta_P \in \Lambda_1$, then the action of g in the covering group is defined by $g \cdot P_p(f) = P_p(g^{-1} \cdot f)$. Then $z_P \mapsto F_g(z_P, \theta_P)$ and $\theta_P \mapsto \Psi_g(z_P, \theta_P)$. So Λ-points transform as the coordinates under the covering group.

Next consider irreducible divisors $P_d = z - \hat{z}_1 - \theta\hat{\theta}_1$, $Q_d = z - \hat{z}_2 - \theta\hat{\theta}_2$. We say that $g \cdot P_d = Q_d$ as divisors if we have the identity $g^{-1} Q_d = P_d h(z, \theta)$ as holomorphic functions for some invertible $h(z, \theta)$. By the same calculation as the one following Lemma 2.2.4 we find that

$$\begin{aligned}
\hat{z}_2 &= F_g(\hat{z}_1, \hat{\theta}_1) + \frac{DF_g(\hat{z}_1, \hat{\theta}_1)}{D\Psi_g(\hat{z}_1, \hat{\theta}_1)} \Psi_g(\hat{z}_1, \hat{\theta}_1), \\
\hat{\theta}_2 &= \frac{DF_g(\hat{z}_1, \hat{\theta}_1)}{D\Psi_g(\hat{z}_1, \hat{\theta}_1)}.
\end{aligned}$$

So irreducible divisors transform with the dual action, compare with (2.9).

There is a parallel theory for the dual curve: we have a covering $(D, \mathcal{O}_{D^{1|1}}) \to (X, \hat{\mathcal{O}}_X)$, with covering group $\hat{G}$. The dual covering group $\hat{G}$ is isomorphic to G by a distinguished isomorphism: g and $\hat{g}$ are identified if they give the same transformation of the reduced disk. Their action on functions is in general different, however, unless we are dealing with a super Riemann surface. In fact, since duality interchanges irreducible divisors and Λ-points on the curve and its dual we see that the action of $\hat{g}$ on the coordinates is dual to the transformation of g:

$$\hat{g}^{-1}\cdot z = F_g(z,\theta) + \frac{DF_g(z,\theta)}{D\Psi_g(z,\theta)}\Psi_g(z,\theta),$$
$$\hat{g}^{-1}\cdot \theta = \frac{DF_g(z,\theta)}{D\Psi_g(z,\theta)}.$$

A function f on $(X,\mathcal{O}_X)$ lifts to a function that is invariant under the covering group G and similarly $\hat{f}$, a function on $(X,\hat{\mathcal{O}}_X)$, lifts to a function that is invariant under the dual covering group $\hat{G}$. An irreducible divisor or a Λ-point on $(X,\mathcal{O}_X)$ lifts to an infinite set of divisors or points, one for each point on the underlying disk above the corresponding reduced point of X.

Let as before x_0 be a point on X and d_0 a point on the disk lying over x_0. Let γ be a contour for integration on $(X,\mathcal{O}_X)$, so γ consists of a contour on X and two irreducible divisors at the endpoints. The contour lifts to a unique contour on the disk starting at d_0 and the irreducible divisors lift to unique irreducible divisors for $(D,\mathcal{O}_{D^{1|1}})$ that reduce to d_0 and the endpoint on the disk, respectively. Also we can pull back sections of $\mathcal{B}\mathrm{er}_X$ to $(D,\mathcal{O}_{D^{1|1}})$ and calculate integrals on $(X,\mathcal{O}_X)$ by lifting to $(D,\mathcal{O}_{D^{1|1}})$. Since D is simply connected this is a great simplification. For instance any integral over a closed contour is zero.

Similar considerations apply to the $N=2$ curve $(X,\mathcal{CO}_X)$ and its universal covering space $D^{1|2}$ and covering group $\mathcal{G}$. Of course $D^{1|2}$ is the $N=2$ curve canonically associated to the $N=1$ curve $D^{1|1}$ as in subsection 2.2, and the lifts of $f\in\mathcal{O}_X$ to $D^{1|2}$ via either $(X,\mathcal{CO}_X)$ or $D^{1|1}$ as intermediate space coincide.

2.6. **Sheaf cohomology for super curves.** Our super curves are in fact families of curves over the base scheme $(\bullet,\Lambda)$, with Λ the Grassmann algebra of nilpotent constants. This means that for any coherent sheaf the cohomology groups are finitely generated Λ-modules, but they are not necessarily free. This means in particular that standard classical theorems, like the Riemann-Roch theorem, do not hold in general in our situation. (See for instance [Hod89].)

The basic facts about sheaf cohomology of families of super curves are completely parallel to the classical theory (explained for instance in [Kem83]). For a coherent locally free sheaf $\mathcal{L}$ there exist Λ-homomorphisms $\alpha: F\to G$, with F,G free finite rank Λ-modules, that calculate the cohomology. More precisely, for every Λ-module M we have an exact sequence

$$0 \to H^0(X, \mathcal{L} \otimes M) \to F \otimes M \overset{\alpha \otimes 1_M}{\to} G \otimes M \to H^1(X, \mathcal{L} \otimes M) \to 0. \tag{2.19}$$

Recall from Example 2.1.1 that for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have an associated split sheaf $\mathcal{F}^{\mathrm{split}} = \mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m}$. Therefore, if we choose $M = \Lambda/\mathfrak{m}$, the sequence (2.19) calculates the cohomology groups of the split sheaf $\mathcal{L}^{\mathrm{split}}$. (These cohomology groups are $\mathbb{Z}_2$-graded vector spaces over $\Lambda/\mathfrak{m} = \mathbb{C}$.) Without loss of generality one can choose the homomorphism $\alpha : F \to G$ such that $\alpha^{\mathrm{split}} = \alpha \otimes 1_{\Lambda/\mathfrak{m}}$ is identically zero. This means that $H^0(X, \mathcal{L})$ (respectively $H^1(X, \mathcal{L})$) is a submodule (resp. a quotient module) of a free Λ-module of rank $\dim H^0(X, \mathcal{L}^{\mathrm{split}})$ (resp. of rank $\dim H^1(X, \mathcal{L}^{\mathrm{split}})$).

We are interested in the question when the $H^i(X, \mathcal{L})$ are free. The idea is to check this by an inductive procedure, starting with the free cohomology of $\mathcal{L}^{\mathrm{split}}$. We have for every $j = 1, \dots, n-1$ the split exact sequence

$$0 \to \mathfrak{m}^j/\mathfrak{m}^{j+1} \to \Lambda/\mathfrak{m}^{j+1} \to \Lambda/\mathfrak{m}^j \to 0. \tag{2.20}$$

Since $\mathfrak{m}^j/\mathfrak{m}^{j+1} \otimes_\Lambda \mathcal{L} = \mathfrak{m}^j/\mathfrak{m}^{j+1} \otimes_{\mathbb{C}} \mathcal{L}^{\mathrm{split}}$, $\Lambda/\mathfrak{m}^i \otimes_\Lambda \mathcal{L} = \mathcal{L}/\mathfrak{m}^i\mathcal{L}$ and $\mathcal{L}$ is flat over Λ we obtain by tensoring with $\mathcal{L}$ and taking cohomology the exact sequence ($\Lambda^j = \mathfrak{m}^j/\mathfrak{m}^{j+1}$)

(2.21)

$$0 \to \Lambda^j \otimes_{\mathbb{C}} H^0(X, \mathcal{L}^{\mathrm{split}}) \to H^0(X, \mathcal{L}/\mathfrak{m}^{j+1}\mathcal{L}) \to H^0(X, \mathcal{L}/\mathfrak{m}^j\mathcal{L}) \overset{q^j}{\to}$$
$$\overset{q^j}{\to} \Lambda^j \otimes_{\mathbb{C}} H^1(X, \mathcal{L}^{\mathrm{split}}) \to H^1(X, \mathcal{L}/\mathfrak{m}^{j+1}\mathcal{L}) \to H^1(X, \mathcal{L}/\mathfrak{m}^j\mathcal{L}) \to 0.$$

If $H^0(X, \mathcal{L}/\mathfrak{m}^j\mathcal{L})$ and $H^1(X, \mathcal{L}/\mathfrak{m}^j\mathcal{L})$ are free $\Lambda/\mathfrak{m}^j$-modules, then the module $H^0(X, \mathcal{L}/\mathfrak{m}^{j+1}\mathcal{L})$ is free over $\Lambda/\mathfrak{m}^{j+1}$ iff the connecting map q^j in (2.21) is zero iff $H^1(X, \mathcal{L}/\mathfrak{m}^{j+1}\mathcal{L})$ is free as $\Lambda/\mathfrak{m}^{j+1}$-module (see [Kem83], Lemma 10.4). The relation between the connecting homomorphisms q^j and the homomorphism α that calculates cohomology is as follows: if we assume as above α^{split} is zero then $q^1 = \alpha \otimes 1_{\Lambda/\mathfrak{m}^2}$. More generally, if $q^1 = q^2 = \cdots = q^{j-1} = 0$ then $q^j = \alpha \otimes 1_{\Lambda/\mathfrak{m}^{j+1}}$.

More concretely, we can assume that α is a matrix of size rank $G \times$ rank F and the q^j are quotients of this matrix by $\mathfrak{m}^{j+1}$. Then the cohomology of $\mathcal{L}$ is the kernel and cokernel of the matrix α, and the cohomology is free iff α is identically zero.

If now $\mathcal{L}$ is an invertible sheaf, $\mathcal{L}^{\text{split}}$ obeys a super Riemann-Roch relation and in case of free cohomology we get ($h^i = \text{rank } H^i$):

(2.22)

$$h^0(X,\mathcal{L}) - h^1(X,\mathcal{L}) = (\deg\mathcal{L} + 1 - g \mid \deg\mathcal{L} + \deg\mathcal{N} + 1 - g),$$

where $\mathcal{O}_X^{\text{split}} = \mathcal{O}_X^{\text{red}} \mid \mathcal{N}$. We can relate by Serre duality the cohomology groups of $\mathcal{L}$ and $\mathcal{L}^* \otimes \mathcal{B}\text{er}_X$, see Appendix A. In particular, $H^0(X, \mathcal{L}^* \otimes \mathcal{B}\text{er}_X)$ is free iff $H^1(X,\mathcal{L})$ is.

We summarize the discussion in this subsection in the following theorem.

Theorem 2.6.1. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-sheaf. Then $H^0(X,\mathcal{L})$ (respectively $H^1(X,\mathcal{L})$) is a submodule (respectively a quotient module) of a free Λ-module of rank* $\dim H^0(X,\mathcal{L}^{\textit{split}})$ *(respectively of rank* $\dim H^1(X,\mathcal{L}^{\textit{split}})$*). Furthermore*

$$\begin{aligned} H^0(X,\mathcal{L}) \text{ is a free } \Lambda\text{-module} &\Longleftrightarrow H^1(X,\mathcal{L}) \text{ is free},\\ &\Longleftrightarrow H^0(X,\mathcal{L}^*\otimes\mathcal{B}er_X) \text{ is free},\\ &\Longleftrightarrow H^1(X,\mathcal{L}^*\otimes\mathcal{B}er_X) \text{ is free}, \end{aligned}$$

in which case the rank of $H^i(X,\mathcal{L})$ is equal to $\dim H^i(X,\mathcal{L}^{\textit{split}})$.

2.7. **Generic SKP curves.**

Definition 2.7.1. An *SKP curve* is a super curve $(X,\mathcal{O}_X)$ such that the split sheaf $\mathcal{O}_X^{\text{split}}$ is of the form

$$\mathcal{O}_X^{\text{split}} = \mathcal{O}_X^{\text{red}} \mid \mathcal{N},$$

where $\mathcal{N}$ is an invertible $\mathcal{O}_X^{\text{red}}$-module of degree zero. If $\mathcal{N} \neq \mathcal{O}_X^{\text{red}}$ then $(X,\mathcal{O}_X)$ is called a *generic SKP curve.* □

We will discuss in subsection 4.1 a Krichever map that associates to an invertible sheaf on a super curve $(X,\mathcal{O}_X)$ (and additional data) a point W of an infinite super Grassmannian. If this point W belongs to the *big cell* (to be defined below) we obtain a solution of the super KP hierarchy. For W to belong to the big cell it is necessary that $(X,\mathcal{O}_X)$ is an SKP curve. The generic SKP curves enjoy simple cohomological properties.

Theorem 2.7.2. *Let $(X,\mathcal{O}_X)$ be a generic SKP curve. Then the cohomology groups of the sheaves $\mathcal{O}_X, \mathcal{B}er_X$ are free Λ-modules. More precisely:*

$$H^0(X,\mathcal{O}_X) = \Lambda \mid 0, \qquad H^1(X,\mathcal{O}_X) = \Lambda^g \mid \Lambda^{g-1},$$
$$H^0(X,\mathcal{B}er_X) = \Lambda^{g-1} \mid \Lambda^g, \qquad H^1(X,\mathcal{B}er_X) = 0 \mid \Lambda.$$

Proof. Since $\mathcal{N}$ has no global sections, $H^0(X, \mathcal{O}_X^{\text{split}}) = \mathbb{C} \mid 0$ consists of the constants only. Now by definition of a curve over $(\bullet, \Lambda)$ we have an inclusion $0 \to \Lambda \to \mathcal{O}_X$, so $H^0(X, \mathcal{O}_X)$ contains at least the constants Λ. By Theorem 2.6.1 then $H^0(X, \mathcal{O}_X)$ must be equal to $\Lambda \mid 0$. Again using Theorem 2.6.1 then also $H^1(X, \mathcal{O}_X)$ and the cohomology of $\mathcal{B}\text{er}_X$ will be free, and the rest of the theorem follows from the properties of the split sheaves, see Examples 2.1.1, 2.2.2. □

Remark 2.7.3. It is not true that all invertible $\mathcal{O}_X$-sheaves for a generic SKP curve have free cohomology. For instance, consider a sheaf $\mathcal{L}$ with $\mathcal{L}^{\text{split}} = \mathcal{O}_X^{\text{split}}$, but $\mathcal{L} \neq \mathcal{O}_X$. Then, for a covering $\{U_\alpha\}$ of X, the transition functions of $\mathcal{L}$ will have the form $g_{\alpha\beta} = 1 + f_{\alpha\beta}(z, \theta)$, with $f_{\alpha\beta}(z, \theta) = 0$ modulo the maximal ideal $\mathfrak{m}$ of Λ. Let then $I \subset \Lambda$ be the ideal of elements that annihilate all $f_{\alpha\beta}$. Then we have $H^0(X, \mathcal{L}) = I$ and is in particular not free. □

2.8. **Riemann bilinear relations.** Let us call sections of $\mathcal{B}\text{er}_X$ and $\mathcal{B}\hat{\text{e}}\text{r}_X$ holomorphic differentials (on $(X, \mathcal{O}_X)$ and $(X, \hat{\mathcal{O}}_X)$ respectively). We will in this subsection introduce analogs of the classical bilinear relations for holomorphic differentials.

Theorem 2.8.1. *Let $(X, \mathcal{O}_X)$ be a super curve and let ω, $\hat{\omega}$ be holomorphic differentials on $(X, \mathcal{O}_X)$ and $(X, \hat{\mathcal{O}}_X)$ respectively. Let $\{a_i, b_i\}$ be a standard symplectic basis for $H_1(X, \mathbb{Z})$. Then*

$$\sum_{i=1}^{g} \oint_{a_i} \omega \oint_{b_i} \hat{\omega} = \sum_{i=1}^{g} \oint_{a_i} \hat{\omega} \oint_{b_i} \omega.$$

Note that we think here of closed contours on the underlying topological space X as closed super contours on either $(X, \mathcal{O}_X)$ or on $(X, \hat{\mathcal{O}}_X)$.

Proof. The argument is clearest using the $N = 2$ curve $(X, \mathcal{C}\mathcal{O}_X)$ and its universal covering superdisk $D^{1|2}$; this way only one universal covering group $\mathcal{G}$ appears instead of both G and $\hat{G}$. Choose any holomorphic differentials ω on X and $\hat{\omega}$ on $\hat{X}$, and lift them to sections $(\omega, 0)$ and $(0, \hat{\omega})$ of $\mathcal{M}$ on $(X, \mathcal{C}\mathcal{O}_X)$. Lifting further to $D^{1|2}$, let Ω be an antiderivative of $(0, \hat{\omega})$, so that $(D_{\mathcal{C}}\Omega, \hat{D}_{\mathcal{C}}\Omega) = (0, \hat{\omega})$. The crucial point is that $(\Omega\omega, 0)$ is itself a section of $\mathcal{M}$, because $D_{\mathcal{C}}(\Omega\omega) = 0$. This could not have been achieved using only differentials from X. As per the standard argument, we integrate this object around the polygon obtained by cutting open $(X, \mathcal{C}\mathcal{O}_X)$. To form this polygon, fix arbitrarily one vertex P (a Λ-point of the $N = 2$ disk $D^{1|2}$) and let the other vertices be $a_1^{-1}P$, $b_1^{-1}a_1^{-1}P, \dots, a_g b_g^{-1} a_g^{-1} \cdots b_1 a_1 b_1^{-1} a_1^{-1} P$, where a_i, b_i are the generating elements of $\mathcal{G}$. The vertices are the endpoints of super contours whose reduced contours are the sides of the usual polygon bounding a fundamental region for $\mathcal{G}$.

These contours project down to any of $X, \hat{X}, (X, \mathcal{C}\mathcal{O}_X)$ as closed loops generating the homology; integrating a differential lifted from any of these spaces along a side of our polygon will yield the corresponding period. Labeling the sides of the polygon with generators of $\mathcal{G}$ as usual, neighborhoods of the sides labeled a_i are identified with each other by b_i and vice versa. Then we have

$$0 = \oint \Omega(\omega, 0) = \sum_{i=1}^{g} \left[\int_{a_i} \Omega(\omega, 0) - \int_{a_i'} \Omega(\omega, 0) \right] + \\ + \sum_{i=1}^{g} \left[\int_{b_i} \Omega(\omega, 0) - \int_{b_i'} \Omega(\omega, 0) \right].$$

In the first sum, the two integrals are related by the change of variables given by b_i; the differential $(\omega, 0)$ is invariant under this covering transformation while Ω changes by the b_i-period of $\hat{\omega}$. The second sum is simplified in the same manner, with the result

$$\sum_{i=1}^{g} \left[\int_{a_i} \omega \int_{b_i} \hat{\omega} - \int_{a_i} \hat{\omega} \int_{b_i} \omega \right] = 0.$$

□

2.9. **The period map and cohomology.** The commutative diagram (2.15) gives a commutative diagram in cohomology that partly reads:

$$\begin{array}{ccccc}
 & & H^0(X, \mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X) & = & H^0(X, \mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X) \\
 & & \downarrow{\scriptstyle \mathrm{p\hat{e}r}} & & \downarrow{\scriptstyle \hat{q}} \\
H^0(X, \mathcal{B}\mathrm{er}_X) & \xrightarrow{\mathrm{per}} & H^1(X, \Lambda) & \xrightarrow{\mathrm{r\hat{e}p}} & H^1(X, \hat{\mathcal{O}}_X) \\
\| & & \downarrow{\scriptstyle \mathrm{rep}} & & \\
H^0(X, \mathcal{B}\mathrm{er}_X) & \xrightarrow{q} & H^1(X, \mathcal{O}_X) & &
\end{array} \tag{2.23}$$

Let $\{a_i, b_i; i = 1, \ldots, g\}$ be a symplectic basis for $H_1(X, \mathbb{Z})$ and let $\{a_i^*, b_i^*; i = 1, \ldots, g\}$ be a dual basis for $H^1(X, \mathbb{Z})$ and also for $H^1(X, \Lambda)$. We will use Serre duality (see Appendix A.2) to identify $H^1(X, \mathcal{O}_X)$ and $H^1(X, \hat{\mathcal{O}}_X)$ with the duals of $H^0(X, \mathcal{B}\mathrm{er}_X)$ and $H^0(X, \mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X)$.

Lemma 2.9.1. *The maps* per*,* pêr*,* rep *and* rêp *are explicitly given by*

$$\mathrm{per}(\omega) = \sum_{i=1}^{g} (\oint_{a_i} \omega) a_i^* + \sum_{i=1}^{g} (\oint_{b_i} \omega) b_i^*,$$

$$\hat{\mathrm{per}}(\hat\omega) = \sum_{i=1}^{g} (\oint_{a_i} \hat\omega) a_i^* + \sum_{i=1}^{g} (\oint_{b_i} \hat\omega) b_i^*,$$

$$\mathrm{rep}(\sigma)(\omega) = \sum_{i=1}^{g} \alpha_i (\oint_{b_i} \omega) - \sum_{i=1}^{g} \beta_i (\oint_{a_i} \omega),$$

$$\hat{\mathrm{rep}}(\sigma)(\hat\omega) = \sum_{i=1}^{g} \alpha_i (\oint_{b_i} \hat\omega) - \sum_{i=1}^{g} \beta_i (\oint_{a_i} \hat\omega),$$

where $\omega \in H^0(X, \mathcal{B}er_X)$*,* $\hat\omega \in H^0(X, \mathcal{B}\hat{er}_X)$ *and* $\sigma = \sum_{i=1}^{g} \alpha_i a_i^* + \beta_i b_i^* \in H^1(X, \Lambda)$*.*

If we introduce a basis $\{\omega_\alpha, \alpha = 1, \dots, g-1 \mid w_j, j = 1, \dots, g\}$ of $H^0(X, \mathcal{B}\mathrm{er}_X)$ we obtain the *period matrix* associated to per:

$$\Pi = \begin{pmatrix} \oint_{a_i} \omega_\alpha & \oint_{a_i} w_j \\ \oint_{b_i} \omega_\alpha & \oint_{b_i} w_j \end{pmatrix},$$

where i, j run from 1 to g and α runs from 1 to $g-1$.

For the split curve we have $H^0(X, \hat{\mathcal{O}}_X^{\mathrm{split}}) = \mathbb{C} \mid \mathbb{C}^{g-1}$ and the map

$$D : H^0(X, \hat{\mathcal{O}}_X^{\mathrm{split}}) \to H^0(X, \mathcal{B}\mathrm{er}_X^{\mathrm{split}})$$

has as image a $g-1$ dimensional, even subspace of exact differentials. For these elements the periods vanish, and one finds the reduction mod $\mathfrak{m}$ of Π is given by

$$\Pi^{\mathrm{split}} = \begin{pmatrix} 0 & \Pi^{\mathrm{red}}(a) \\ 0 & \Pi^{\mathrm{red}}(b) \end{pmatrix},$$

where $\Pi^{\mathrm{red}} = \begin{pmatrix} \Pi^{\mathrm{red}}(a) \\ \Pi^{\mathrm{red}}(b) \end{pmatrix}$ is the classical period matrix of the underlying curve $(X, \mathcal{O}_X^{\mathrm{red}})$. By classical results we can choose the basis of holomorphic differentials on the reduced curve so that $\Pi^{\mathrm{red}}(a) = 1_g$. From this it follows that we can also choose in $H^0(X, \mathcal{B}\mathrm{er}_X)$ a basis such that $\oint_{a_i} w_j = \delta_{ij}$ and so that the period matrix takes the form

$$\Pi = \begin{pmatrix} 0 & 1_g \\ Z_o & Z_e \end{pmatrix}. \tag{2.24}$$

Note that Π is not uniquely determined by the conditions we have imposed: we are still allowed to change $\Pi \mapsto \Pi' = \begin{pmatrix} 0 & 1_g \\ Z_o G & Z_e + Z_o \Gamma \end{pmatrix}$, corresponding to a change of basis of $H^0(X, \mathcal{B}\mathrm{er}_X)$ by an even invertible matrix $\begin{pmatrix} G & \Gamma \\ 0 & 1_g \end{pmatrix}$ of size $g-1 \mid g \times g-1 \mid g$.

Using the same basis we see that rep has matrix

$$\begin{pmatrix} \oint_{b_i} \omega_\alpha & -\oint_{a_i} \omega_\alpha \\ \oint_{b_i} w_j & -\oint_{a_i} w_j \end{pmatrix} = \Pi^t I = \begin{pmatrix} Z_o^t & 0 \\ Z_e^t & -I_g \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}.$$

Again this matrix is not entirely determined by our choices.

From the commutativity of the diagram (2.23) we see that the matrix of the map q is given by

$$Q = \Pi^t I \Pi = \begin{pmatrix} 0 & Z_o^t \\ -Z_o & Z_e^t - Z_e \end{pmatrix}. \tag{2.25}$$

In general, the structure sheaf $\hat{\mathcal{O}}_X$ and dualizing sheaf $\mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X$ of the dual curve will not have free cohomology, so that we cannot represent the maps rep, rêp and $\hat{q}$ by explicit matrices.

The nonfreeness of the cohomology of $\hat{\mathcal{O}}_X$ and $\mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X$ is determined by the odd component Z_o of the period matrix, see (2.24). Recall that $\hat{\mathcal{O}}_X^{\mathrm{split}} = \mathcal{O}_X^{\mathrm{red}} \mid \mathcal{K}\mathcal{N}^{-1}$ and $\mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X^{\mathrm{split}} = \mathcal{N} \mid \mathcal{K}$ (see Example 2.2.3) and hence

$$H^0(X, \hat{\mathcal{O}}_X^{\mathrm{split}}) = \mathbb{C} \mid \mathbb{C}^{g-1}, \quad H^1(X, \hat{\mathcal{O}}_X^{\mathrm{split}}) = \mathbb{C}^g \mid 0,$$
$$H^0(X, \mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X^{\mathrm{split}}) = 0 \mid \mathbb{C}^g, \quad H^1(X, \mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X^{\mathrm{split}}) = \mathbb{C}^{g-1} \mid \mathbb{C}.$$

From the diagram (2.15) we extract in cohomology, using that the map $H^1(X, \mathcal{B}\mathrm{er}_X) \to H^2(X, \Lambda) = \Lambda \mid 0$ is an (odd!) isomorphism and $H^0(X, \Lambda) = \Lambda$,

$$0 \to \frac{H^0(X, \hat{\mathcal{O}}_X)}{\Lambda} \xrightarrow{D} H^0(X, \mathcal{B}\mathrm{er}_X) \xrightarrow{\mathrm{per}} H^1(X, \Lambda) \to H^1(X, \hat{\mathcal{O}}_X) \to 0 \tag{2.26}$$

so that the period map has as kernel $H^0(X, \hat{\mathcal{O}}_X)$ mod constants and as cokernel $H^1(X, \hat{\mathcal{O}}_X)$. Therefore per is essentially one of the homomorphisms that calculate cohomology introduced in subsection 2.6. We can even be more explicit: if $\{\omega_\alpha \mid w_j\}$ is the (partially) normalized basis of holomorphic differentials as above the homomorphism per maps the submodule generated by the w_j isomorphically to a free rank g summand of $H^1(X, \Lambda)$. This is irrelevant for the calculation of cohomology, so we can replace the sequence (2.26) by

$$0 \to H^0(X, \hat{\mathcal{O}}_X)/\Lambda \xrightarrow{D} \Lambda^{g-1} \xrightarrow{Z_o} \Lambda^g \to H^1(X, \hat{\mathcal{O}}_X) \to 0, \tag{2.27}$$

and $H^0(X,\hat{\mathcal{O}}_X)$ mod constants is the kernel of Z_o, whereas the cokernel of Z_o is $H^1(X,\hat{\mathcal{O}}_X)$.

Similarly, the cohomology of $\mathcal{B}\hat{\text{e}}\text{r}_X$ is calculated by the sequence

$$\begin{aligned}(2.28)\quad 0\to H^0(X,\mathcal{B}\hat{\text{e}}\text{r}_X)\overset{\text{p}\hat{\text{e}}\text{r}}{\to} H^1(X,\Lambda)\overset{\text{rep}}{\to} H^0(X,\mathcal{B}\text{er}_X)^*\\ \to H^1(X,\mathcal{B}\hat{\text{e}}\text{r}_X)\to\Lambda\to 0\end{aligned}$$

The image of a holomorphic differential $\hat\omega$ in $H^1(X,\Lambda)$ is then a vector $\text{p}\hat{\text{e}}\text{r}(\hat\omega)=\begin{pmatrix}a(\hat\omega)\\ b(\hat\omega)\end{pmatrix}$, where $a(\hat\omega)$ and $b(\hat\omega)$ are the vectors of a respectively b periods of $\hat\omega$. By exactness of (2.28) we have $\text{rep}\circ\text{p}\hat{\text{e}}\text{r}=0$, or, using bases,

$$\begin{pmatrix}Z_o^t & 0\\ Z_e^t & -I_g\end{pmatrix}\begin{pmatrix}a(\hat\omega)\\ b(\hat\omega)\end{pmatrix}=0$$

This means that the vector $b(\hat\omega)$ of b periods is (uniquely) determined by the a periods: $b(\hat\omega)=Z_e^t a(\hat\omega)$, and the vector of a periods is constrained by the equation $Z_o^t a(\hat\omega)=0$. The submodule of $H^1(X,\Lambda)$ generated by the elements b_i^* maps under rep isomorphically to a free rank $0\mid g$ summand of $H^1(X,\mathcal{B}\text{er}_X)^*$, so that for the calculation of cohomology we can simplify (2.28) to

$$(2.29)\quad 0\to H^0(X,\mathcal{B}\hat{\text{e}}\text{r}_X)\to\Lambda^g\overset{Z_o^t}{\to}\Lambda^{g-1}\to H^1(X,\mathcal{B}\hat{\text{e}}\text{r}_X)\to\Lambda\to 0.$$

We summarize the results on the cohomology of the dual curve in the following Theorem.

Theorem 2.9.2. *Let $(X,\mathcal{O}_X)$ be a generic SKP curve with odd period matrix Z_o. Then*

$$H^0(X,\hat{\mathcal{O}}_X)/\Lambda\simeq\text{Ker}(Z_o),\quad H^1(X,\hat{\mathcal{O}}_X)\simeq\text{Coker}(Z_o).$$

Furthermore $H^0(X,\mathcal{B}\hat{er}_X)\simeq\text{Ker}(Z_o^t)$ and $\text{Coker}(Z_o^t)$ is a submodule of $H^1(X,\mathcal{B}\hat{er}_X)$ such that

$$H^1(X,\mathcal{B}\hat{er}_X)/\text{Coker}(Z_o^t)\simeq\Lambda.$$

2.10. **$\mathcal{CO}_X$ as extension of $\mathcal{B}\text{er}_X$.** We discuss in this subsection, for generic SKP curves, the structure of $\mathcal{CO}_X$ as extension of $\mathcal{B}\text{er}_X$ and the relation with free cohomology and the projectedness of the curve $(X,\mathcal{O}_X)$.

From the sequence (2.7) that defines $\mathcal{CO}_X$ we obtain in cohomology

$$\begin{aligned}(2.30)\quad 0\to H^0(X,\mathcal{O}_X)\to H^0(X,\mathcal{CO}_X)\to H^0(X,\mathcal{B}\text{er}_X)\overset{q}{\to}\\ \overset{q}{\to} H^1(X,\mathcal{O}_X)\to H^1(X,\mathcal{CO}_X)\to H^1(X,\mathcal{B}\text{er}_X)\to 0\end{aligned}$$

The cohomology of the sheaves $\mathcal{O}_X, \mathcal{B}\mathrm{er}_X$ is given by Theorem 2.7.2. By Theorem 2.6.1 (or its extension to rank two sheaves) $H^0(X, \mathcal{C}\mathcal{O}_X)$ is a submodule of a $\Lambda^{g+1} \mid \Lambda^{g-1}$ and $H^1(X, \mathcal{C}\mathcal{O}_X)$ is a quotient of a $\Lambda^{g+1} \mid \Lambda^{g-1}$. We see from this that the cohomology of $\mathcal{C}\mathcal{O}_X$ is free if and only if q is the zero map.

To describe the map q in more detail we need to recall some facts about principal parts and extensions, (see e.g., [Kem83]). For any invertible sheaf $\mathcal{L}$ let $\underline{\mathcal{R}at}(\mathcal{L})$ and $\underline{\mathcal{P}rin}(\mathcal{L})$ denote the sheaves of rational sections and principal parts for $\mathcal{L}$ and denote by $\mathcal{R}at(\mathcal{L})$ and $\mathcal{P}rin(\mathcal{L})$ their Λ-modules of global sections. Then the cohomology of $\mathcal{L}$ is calculated by

$$0 \to H^0(X, \mathcal{L}) \to \mathcal{R}at(\mathcal{L}) \to \mathcal{P}rin(\mathcal{L}) \to H^1(X, \mathcal{L}) \to 0.$$

In particular we can represent a class $\alpha \in H^1(X, \mathcal{L})$ as a principal part $p = \sum p_x$, where $p_x \in \mathcal{R}at(\mathcal{L})/\mathcal{L}_x$, for $x \in X$.

If $\alpha \in H^1(X, \mathcal{L})$ and $\omega \in H^0(X, \mathcal{M})$, for some other invertible sheaf $\mathcal{M}$, then we can define the *cup product* $\omega \cup \alpha$ by representing α by a principal part p and calculating the principal part $\omega p = \sum \omega_x p_x$ in $\mathcal{P}rin(\mathcal{M} \otimes \mathcal{L})$; the image of ωp in $H^1(X, \mathcal{M} \otimes \mathcal{L})$ is then by definition $\omega \cup \alpha$.

We want to understand the kernel of the cup product with $\omega \in H^0(X, \mathcal{M})$ in case ω is odd and free (i.e., linearly independent over Λ). In this case there will be for any invertible sheaf $\mathcal{L}$ sections that are immediately annihilated by ω; let therefore $\operatorname{Ann}(\mathcal{L}, \omega) \subset \mathcal{L}$ be the subsheaf of such sections. Putting $\mathcal{L}_\omega = \mathcal{L}/\operatorname{Ann}(\mathcal{L}, \omega)$, we get, because $\omega^2 = 0$, the exact sequence

$$0 \to \mathcal{L}_\omega \xrightarrow{\omega} \operatorname{Ann}(\mathcal{L} \otimes \mathcal{M}, \omega) \to Q \to 0 \tag{2.31}$$

Locally, in an open set $U_\alpha \subset X$, we have $\mathcal{L}(U_\alpha) = \mathcal{O}_X(U_\alpha) l_\alpha$, $\mathcal{M}(U_\alpha) = \mathcal{O}_X(U_\alpha) m_\alpha$ and we write $\omega = \omega_\alpha(z, \theta) m_\alpha$, with $\omega_\alpha = \phi_\alpha + \theta_\alpha f_\alpha$. Then f_α^{red} is a regular function on U_α with some divisor of zeros $D_f = \sum n_i q_i$. Some of the q_i may also be zeros of (the lowest order part of) ϕ_α and there will be a maximal $g_\alpha(z, \theta) \in \mathcal{O}_X(U_\alpha)_{\bar{0}}$ (here $\mathcal{O}_X(U_\alpha)_{\bar{0}}$ is the module of even sections) such that

$$\omega_\alpha(z, \theta) = \omega_\alpha(z, \theta)' g_\alpha(z, \theta)$$

with g_α^{red} a regular function with divisor of zeros D_g (on U_α) satisfying $0 \le D_g \le D_f$. Then $\operatorname{Ann}(\mathcal{L} \otimes \mathcal{M}, \omega)(U_\alpha)$ is generated by $\omega_\alpha(z, \theta)' l_\alpha \otimes m_\alpha$ and we see that Q is a torsion sheaf: Q is killed by the invertible sheaf generated locally by the even invertible rational function $g_\alpha(z, \theta)$. Let $D_\omega = \{(g_\alpha(z, \theta), U_\alpha)\}$ be the corresponding Cartier divisor. Then we have an isomorphism

$$\mathrm{Ann}(\mathcal{L}\otimes\mathcal{M},\omega)\to\mathcal{L}(D_\omega),\quad \omega_\alpha(z,\theta)'l_\alpha\otimes m_\alpha\mapsto l_\alpha\otimes 1/g_\alpha(z,\theta)$$

The sequence (2.31) is equivalent to

$$0\to\mathcal{L}\to\mathcal{L}(D_\omega)\to\mathcal{L}(D_\omega)|_{D_\omega}\to 0.$$

Now the cup product with ω gives a map $H^1(\mathcal{L}_\omega)\to H^1(X,\mathrm{Ann}(\mathcal{L}\otimes\mathcal{M},\omega))$ with kernel the image of the natural map $\phi: H^0(X,Q)\to H^1(\mathcal{L}_\omega)$. Identifying $H^0(X,Q)$ with $H^0(X,\mathcal{L}(D_\omega)|_{D_\omega})$, we see that ϕ is the composition

$$H^0(X,\mathcal{L}(D_\omega)|_{D_\omega})\to\mathcal{P}rin(\mathcal{L})\to H^1(X,\mathcal{L}).$$

Therefore the kernel of $\omega\cup$ consists of those $\alpha\in H^1(X,\mathcal{L})$ that have a representative $p\in\mathcal{P}rin(\mathcal{L})$ such that ωp has zero principal part, i.e., the poles in p are compensated by the zeros in ω.

Extensions of the form (2.7) are classified by $\delta\in H^1(X,\mathcal{B}\mathrm{er}_X^*)$: we think of $\mathcal{C}\mathcal{O}_X$ as a subsheaf of $\underline{\mathcal{R}at}(\mathcal{O}_X\oplus\mathcal{B}\mathrm{er}_X)$ consisting on an open set U of pairs (f,ω) where $\omega\in\mathcal{B}\mathrm{er}_X(U)$ and f a rational function such that the principal part $\bar f$ is equal to ωp, for $p\in\mathcal{P}rin(\mathcal{B}\mathrm{er}_X^*)$; then $\delta\in H^1(X,\mathcal{B}\mathrm{er}_X^*)$ is the class of p. It is then easy to see that the connecting map $q: H^0(X,\mathcal{B}\mathrm{er}_X)\to H^1(X,\mathcal{O}_X)$ is cup product by the extension class δ: $q(\omega)=\omega\cup\delta$. The class δ is ly represented by the *Čech* cocycle

$$\phi_{\beta\alpha}=\partial_\theta F_{\beta\alpha}/\partial_\theta\Psi_{\beta\alpha}\in\mathcal{B}\mathrm{er}_X^*(U_\alpha\cap U_\beta), \tag{2.32}$$

from (2.3), (2.5).

Lemma 2.10.1. *Let $(X,\mathcal{O}_X)$ be a generic SKP curve. Then the connecting homomorphism $q: H^0(X,\mathcal{B}er_X)\to H^1(X,\mathcal{O}_X)$ in* (2.30) *is the zero map iff the extension* (2.7) *is trivial. In particular the cohomology of $\mathcal{C}\mathcal{O}_X$ is free iff the extension is trivial.*

Proof. It is clear that if the extension is trivial the connecting map q is trivial.

From the explicit form, in particular the θ independence, of the cocycle we see that it is not immediately killed by multiplication by an odd free section ω of $\mathcal{B}\mathrm{er}_X$, i.e., the cocycle is not zero in the cohomology group $H^1(X,\mathcal{B}\mathrm{er}_X^*)/\mathrm{Ann}(\mathcal{B}\mathrm{er}_X^*,\omega)$ if it is nonzero in $H^1(X,\mathcal{B}\mathrm{er}_X^*)$. The split sheaf $\mathcal{B}\mathrm{er}_X^{\mathrm{split}}$ is $\mathcal{K}\mathcal{N}^{-1}\mid\mathcal{K}$. An odd free section ω of $\mathcal{B}\mathrm{er}_X$ therefore has an associated divisor D_ω as constructed above with reduced support included in the divisor of a section of $\mathcal{K}$ on the underlying curve. Now $q(\omega)=\omega\cup\delta$ is zero if the zeros of ω cancel the poles occuring in the principal part p representing δ. But by classical results the complete linear system of $\mathcal{K}$ has no base points, i.e., there is no point on X where where all global sections of $\mathcal{K}$ vanish. This means that wherever the poles of p occur, there will be a section ω of $\mathcal{B}\mathrm{er}_X$ that does not vanish there. So q being zero on all odd generators of $H^0(X, \mathcal{B}\mathrm{er}_X)$ implies that the extension is trivial. A fortiori if q is the zero map the extension will also be trivial. □

The extension given by the cocycle (2.32) is trivial if

$$\phi_{\beta\alpha}(z_\alpha) = \sigma_\beta(z_\beta, \theta_\beta) - H_{\beta\alpha}(z_\alpha, \theta_\alpha)\sigma_\alpha(z_\alpha, \theta_\alpha) \tag{2.33}$$

for some 1-cochain $\sigma_\alpha \in \mathcal{B}\mathrm{er}^*_X(U_\alpha)$. In that case, a splitting $e : \mathcal{B}\mathrm{er}_X \to \mathcal{C}\mathcal{O}_X$ is obtained by $e(f_\alpha) = \rho_\alpha - \sigma_\alpha(z_\alpha, \theta_\alpha)$.

Theorem 2.10.2. *For a generic SKP curve $(X, \mathcal{O}_X)$, $\mathcal{C}\mathcal{O}_X$ is a trivial extension of $\mathcal{B}er_X$ iff $(X, \mathcal{O}_X)$ is projected.*

Proof. We have already observed (in subsection 2.2) that X projected implies $\phi_{\beta\alpha} = 0$ in a projected atlas, making the extension trivial.

Now suppose, if possible, that the extension is trivial but that X is not projected. Write the transition functions of X in the form

$$z_\beta = f_{\beta\alpha}(z_\alpha) + \theta_\alpha \eta_{\beta\alpha}(z_\alpha), \quad \theta_\beta = \psi_{\beta\alpha}(z_\alpha) + \theta_\alpha B_{\beta\alpha}(z_\alpha)$$

and assume that the atlas has been chosen so that $\eta_{\beta\alpha}$ vanishes to the highest possible (odd) order n in nilpotents. That is, $\eta_{\beta\alpha} = 0 \bmod \mathfrak{m}^n$, but not mod $\mathfrak{m}^{n+2}$. Writing also $\sigma_\alpha(z_\alpha, \theta_\alpha) = \chi_\alpha(z_\alpha) + \theta_\alpha h_\alpha(z_\alpha)$ and substituting in (2.33) yields two conditions. From the θ_α-independence of $\phi_{\beta\alpha}$ one finds that $h_\alpha \bmod \mathfrak{m}^{n+1}$ is a global section of $\mathcal{K}^{-1}\mathcal{N}^2$. Since X is a generic SKP curve, there are no such sections and $h_\alpha = 0$ to this order. Using this, the second condition becomes,

$$\eta_{\beta\alpha} = B_{\beta\alpha}\chi_\beta(f_{\beta\alpha}) - f'_{\beta\alpha}\chi_\alpha \mod \mathfrak{m}^{n+2}.$$

This condition implies that the coordinate change $\tilde{z}_\alpha = z_\alpha - \theta_\alpha\chi_\alpha$ will make $\eta_{\beta\alpha}$ vanish to higher order than n, a contradiction. □

To lowest order in nilpotents, the cocycle conditions for the transition functions of X imply that $\eta_{\beta\alpha}/B_{\beta\alpha}$ is a cocycle for $H^1(X, \mathcal{N}\mathcal{K}^{-1})$, while $\psi_{\beta\alpha}$ is a cocycle for $H^1(X, \mathcal{N}^{-1})$. This implies that the projected X's have codimension $(0 \mid 3g-3)$ in the moduli space of generic SKP curves, which has dimension $(4g-3 \mid 4g-4)$ (see [Vai90]). The proof of Theorem 2.10.2 generalizes to higher order in nilpotents the fact that at lowest order $\phi_{\beta\alpha}$ is a cocycle in $H^1(X, \mathcal{N}\mathcal{K}^{-1} \mid \mathcal{N}^2\mathcal{K}^{-1})$.

2.11. **$\mathcal{C}\mathcal{O}_X$ as extension of $\mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X$ and symmetric period matrices.** One can equally view $\mathcal{C}\mathcal{O}_X$ as an extension of $\mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X$ by $\hat{\mathcal{O}}_X$. Obviously, if $(X, \hat{\mathcal{O}}_X)$ is projected this extension is trivial, but the converse no longer holds. In the proof of Theorem 2.10.2 there is now the possibility that $h_\alpha \neq 0$. (Recall from subsection 2.3 that for X a SRS, a splitting of the extension was universally given by $\chi_\alpha = 0, h_\alpha = -1$.) One can see that this extension is not always trivial, however, by constructing examples with $\psi_{\beta\alpha} = 0$ and $\phi_{\beta\alpha}$ a nontrivial class. (We are now refering to an atlas for $(X, \hat{\mathcal{O}}_X)$.) In this subsection we will exhibit a connection between the structure of $\mathcal{CO}_X$ as extension of $\mathcal{B}\hat{\text{e}}\text{r}_X$ and the symmetry of the component Z_e of the period matrix, see (2.24).

By classical results Z_e^{red} is symmetric. However, there seems to be no reason that Z_e is symmetric in general.

Theorem 2.11.1. *Let $(X, \mathcal{O}_X)$ be a generic SKP curve and Z_e, Z_o its (partially) normalized period matrices (as in (2.24)). Then we have Z_e symmetric and $Z_o = 0$ iff $(X, \mathcal{O}_X)$ is projected.*

Proof. This follows immediately from Theorem 2.10.2, Lemma 2.10.1 and the explicit form (2.25) of the connecting homomorphism q. □

Recall the exact sequence

$$0 \to \Lambda \to \mathcal{CO}_X \overset{(D_\mathcal{C}, \hat{D}_\mathcal{C})}{\to} \mathcal{M} \to 0,$$

where $\mathcal{M} = \mathcal{B}\text{er}_X \oplus \mathcal{B}\hat{\text{e}}\text{r}_X$ is the sheaf of objects that can be integrated on $\mathcal{CO}_X$. The corresponding cohomology sequence is in part

$$0 \to \Lambda \to H^0(X, \mathcal{CO}_X) \overset{(D_\mathcal{C}, \hat{D}_\mathcal{C})}{\longrightarrow} H^0(X, \mathcal{M}) \overset{\text{cper}}{\longrightarrow} H^1(X, \Lambda)$$

where $\text{cper}(\omega, \hat{\omega}) = \{\sigma \mapsto \int_\sigma [\omega + \hat{\omega}]\}$. So we see that we can identify $H^0(X, \mathcal{CO}_X)/\Lambda$ with pairs $(\omega, \hat{\omega})$ of differentials with opposite periods.

Now let $(\omega, \hat{\omega})$ be such a pair. ω can be written in terms of the basis of $H^0(X, \mathcal{B}\text{er}_X)$ in the form

$$\omega = \sum a_i(\omega) w_i + \sum A_\alpha \omega_\alpha,$$

where $a_i(\omega)$ denote the a-periods and A_α are other constants uniquely determined by ω. Then the vector of b-periods of ω will be

$$b(\omega) = Z_e a(\omega) + Z_o A.$$

Since these coincide with minus the b-periods of $\hat{\omega}$, which are $b(\hat{\omega}) = Z_e^t a(\hat{\omega}) = -Z_e^t a(\omega)$, we obtain for each such pair of differentials a relation

$$(Z_e - Z_e^t) a(\omega) + Z_o A = 0. \tag{2.34}$$

We have a sequence analogous to (2.30) for $\mathcal{CO}_X$ as extension of $\mathcal{B}\hat{\text{e}}\text{r}_X$ and a connecting map $\hat{q}$ for this situation.

Theorem 2.11.2. *Let $(X, \mathcal{O}_X)$ be a generic SKP curve and Z_e, Z_o its normalized period matrices. If Z_e is symmetric, then $\hat{q}$ is the zero map.*

Proof. Assuming that $Z_e = Z_e^t$, we determine the set of pairs $(\omega, \hat{\omega})$ with opposite periods. The a-periods of $\hat{\omega}$ can be chosen freely from the kernel of Z_o^t. According to (2.34), any ω chosen to match these a-periods will also have matching b-periods iff A_α belongs to the kernel of Z_o. Therefore, $H^0(X, \mathcal{C}\mathcal{O}_X)$ mod constants can be identified with $\mathrm{Ker} Z_o \oplus \mathrm{Ker} Z_o^t$, which is precisely $H^0(X, \hat{\mathcal{O}}_X)/\Lambda \oplus H^0(X, \mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X)$. In this case $\hat{q}$ is the zero map. □

In general it seems that $\hat{q} = 0$ will not imply that the extension $\mathcal{C}\mathcal{O}_X$ of $\mathcal{B}\hat{\mathrm{e}}\mathrm{r}_X$ is trivial, as in Lemma 2.10.1 for the extension of $\mathcal{B}\mathrm{er}_X$ by $\mathcal{O}_X$. Also it seems that $Z_e = Z_e^t$ cannot be deduced from (2.34) as long as the a-periods are constrained to the kernel of Z_o^t.

2.12. **Moduli of invertible sheaves.** In this subsection we will discuss some facts about invertible sheaves on super curves and their moduli spaces, see also [RSV88, GN88b].

An invertible sheaf on $(X, \mathcal{O}_X)$ is determined by transition functions $g_{\alpha\beta}$ on overlaps $U_\alpha \cap U_\beta$, and so isomorphism classes of invertible sheaves are classified by the cohomology group $H^1(X, \mathcal{O}_{X,\mathrm{ev}}^\times)$.

The degree of an invertible sheaf $\mathcal{L}$ is the degree of the underlying reduced sheaf $\mathcal{L}^{\mathrm{red}}$, with transition functions $g_{\alpha\beta}^{\mathrm{red}}$. Let $\mathrm{Pic}^0(X)$ denote the group of degree zero invertible sheaves on $(X, \mathcal{O}_X)$. The exponential sheaf sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X,\mathrm{ev}} \overset{\exp(2\pi i \times \cdot)}{\to} \mathcal{O}_{X,\mathrm{ev}}^\times \to 0 \tag{2.35}$$

reduces mod nilpotents to the usual exponential sequence for $\mathcal{O}_X^{\mathrm{red}}$ and we see that $\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_{X,\mathrm{ev}})/H^1(X, \mathbb{Z})$.

If $(X, \mathcal{O}_X)$ is a generic SKP curve $H^1(X, \mathcal{O}_X)$ is a free rank $g \mid g-1$ Λ-module and the map $H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X)$ is the restriction of the map $H^1(X, \Lambda) \to H^1(X, \mathcal{O}_X)$, which is dual to the map per of Lemma 2.9.1. So with respect to a suitable basis $H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X)$ is described by the transpose of the period matrix (2.24). This implies that the image of $H^1(X, \mathbb{Z})$ is generated by $2g$ elements that are linearly independent over the real part $\Lambda_\Re$ of Λ (see Appendix B for the definition of $\Lambda_\Re$). The elements of the quotient $\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_{X,\mathrm{ev}})/H^1(X, \mathbb{Z})$ are the Λ-points of a super torus of dimension $(g \mid g-1)$. Each component of $\mathrm{Pic}(X)$ is then isomorphic as a supermanifold to this supertorus.

In general, however, $H^1(X, \mathcal{O}_X)$ is not free, nor is the image of $H^1(X, \mathbb{Z})$ generated by $2g$ independent vectors. It seems an interesting question to understand $\mathrm{Pic}^0(X)$ in this generality.

For any supercurve $(X, \mathcal{O}_X)$ we define the Jacobian by

$$\mathrm{Jac}(X) = H^0(X, \mathcal{B}\mathrm{er}_X)^*_{\mathrm{odd}}/H_1(X, \mathbb{Z}),$$

where elements of $H_1(X, \mathbb{Z})$ act by odd linear functionals on holomorphic differentials from $H^0(X, \mathcal{B}\mathrm{er}_X)$ by integration over 1-cycles.

We have, as discussed in Appendix A, a pairing of Λ-modules

$$H^1(X, \mathcal{O}_X) \times H^0(X, \mathcal{B}\mathrm{er}_X) \to \Lambda. \tag{2.36}$$

As we will discuss in more detail in subsection 2.13 invertible sheaves are also described by divisor classes. We use this in the following Theorem.

Theorem 2.12.1. *The pairing (2.36) induces an isomorphism of the identity component* $\mathrm{Pic}^0(X)$ *with the Jacobian* $\mathrm{Jac}(X)$ *given by the usual Abel map: a bundle* $\mathcal{L} \in \mathrm{Pic}^0(X)$ *with divisor* $P - Q$ *corresponds to the class of linear functionals* $\int_Q^P$*, modulo the action of* $H_1(X, \mathbb{Z})$ *by addition of cycles to the path from* Q *to* P*.*

Proof. Let $\mathcal{L} \in \mathrm{Pic}^0(X)$ have divisor $P - Q$, with the reduced points P^{red} and Q^{red} contained in a single chart U_0 of a good cover of X. If $P = z - p - \theta\pi$ and $Q = z - q - \theta\xi$, this bundle has a canonical section equal to unity in every other chart, and equal to

$$\frac{z - p - \theta\pi}{z - q - \theta\xi} = \frac{z - p}{z - q} - \frac{\theta\pi}{z - q} + \theta\xi\frac{z - p}{(z - q)^2}$$

in U_0. In the covering space $H^1(X, \mathcal{O}_{X,\mathrm{ev}})$ of $\mathrm{Pic}^0(X)$, with covering group $H^1(X, \mathbb{Z})$, $\mathcal{L}$ lifts to a discrete set of cocycles given by the logarithms of the transition functions of $\mathcal{L}$ in the chart overlaps, namely

$$a_{0i} = \frac{1}{2\pi i}[\log(z - p) - \log(z - q) - \frac{\theta\pi}{z - p} + \frac{\theta\xi}{z - q}]$$

in $U_0 \cap U_i$, and zero in other overlaps. The covering group acts by changing the choice of branches for the logarithms. We now fix the particular choice for which the branch cut C from Q to P lies entirely in U_0 and meets no other U_i. Under the Dolbeault isomorphism, this cocycle corresponds to a $(0, 1)$ form most conveniently represented by the current $\bar\partial a_i$ in U_i, where $a_{ij} = a_i - a_j$ and $\bar\partial = d\bar z \partial_{\bar z} + d\bar\theta\partial_{\bar\theta}$. It is supported on the branch cut C, and we can take $a_i = 0$ for $i \neq 0$. The pairing (2.36) now associates to this the linear functional on

$H^0(X,\mathcal{B}\mathrm{er}_X)$ which sends $\omega \in H^0(X,\mathcal{B}\mathrm{er}_X)$, written as $f(z)+\theta\phi(z)$ in U_0, to [HW87]

$$\int_X i(\partial_{\bar z})\bar\partial a_0\,\omega\bar\theta\,[dz\,d\bar z\,d\bar\theta\,d\theta] = \int_X (\partial_{\bar z} a_0)\omega dz\,d\bar z\,d\theta.$$

By the definition of the derivative of a current [GH78] and Stokes' theorem this can be rewritten

$$\begin{aligned}&-\int_{\partial(X-C)} dz \int d\theta\, a_0(f+\theta\phi) =\\ &= -\frac{1}{2\pi i}\int_{\partial(X-C)} dz\{[\log(z-p)-\log(z-q)]\phi + [\frac{\xi}{z-q}-\frac{\pi}{z-p}]f\},\end{aligned}$$

where $\partial(X-C)$ denotes the limit of a small contour enclosing C. Using the residue theorem and the discontinuity of the logarithms across the cut, this evaluates to

$$\int_C \phi\,dz + \pi f(p) - \xi f(q) = \int_Q^P \omega.$$

By linearity of the pairing (2.36), we can extend this correspondence to arbitrary bundles of degree zero by taking sums of divisors of the form $P_i - Q_i$. In particular, the divisor $(P-Q)+(P_1-P)+(P_2-P_1)+\cdots+(P_n-P_{n-1})+(P-P_n)$ is equivalent to $P-Q$, but if the contour $PP_1P_2\cdots P_nP$ represents a nontrivial homology class then the corresponding linear functionals $\int_Q^P$ differ by addition of this cycle to the integration contour. This shows that the action of $H_1(X,\mathbb{Z})$ specified in the definition of $\mathrm{Jac}(X)$ is the correct one. □

2.13. **Effective divisors and Poincaré sheaf for generic SKP curves.** Another description of invertible sheaves is given by divisor classes.

Recall that a divisor $D \in \mathrm{Div}(X)$ is a global section of the sheaf $\mathcal{R}at^\times_{\mathrm{ev}}(X)/\mathcal{O}^\times_{X,\mathrm{ev}}$, so D, up to equivalence, is given by a collection (f_α, U_α) where the f_α are even invertible rational functions that are on overlaps related by an element of $\mathcal{O}^\times_{X,\mathrm{ev}}(U_\alpha \cap U_\beta)$. Each f_α reduces mod nilpotents to a nonzero rational function f_α^{red} on the reduced curve, so that D determines a divisor D^{red}. Then the *degree* of D is the usual degree of its reduction D^{red}. We have a mapping $\mathcal{R}at^\times_{\mathrm{ev}}(X) \to \mathrm{Div}(X)$, $f \mapsto (f)$, and elements (f) of the image are called *principal*. Two even invertible rational functions f_1, f_2 give rise to the same divisor iff $f_1 = kf_2$ where $k \in H^0(X,\mathcal{O}^\times_{X,\mathrm{ev}})$. So if $(X,\mathcal{O}_X)$ is a generic SKP curve k is just an even invertible element of Λ but in general more exotic possibilities for k exist. A divisor D is *effective*, notation

$D \geq 0$, if all $f_\alpha \in \mathcal{O}_{X,\mathrm{ev}}(U_\alpha)$. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ can be thought of as a submodule of rank $1|0$ of the constant sheaf $\mathcal{R}at(X)$. If $\mathcal{L}(U_\alpha) = \mathcal{O}_X(U_\alpha)e_\alpha$, then $e_\alpha \in \mathcal{R}at^\times_{\mathrm{ev}}(X)$ and $\mathcal{L}$ determines the divisor $D = \{(f_\alpha = e_\alpha^{-1}, U_\alpha)\}$. Conversely any divisor D determines an invertible sheaf $\mathcal{O}_X(D)$ (in $\mathcal{R}at(X)$) with local generators $e_\alpha = f_\alpha^{-1}$. Two divisors $D_1 = \{(f_\alpha^{(1)}, U_\alpha)\}$ and $D_2 = \{(f_\alpha^{(2)}, U_\alpha)\}$ give rise to equivalent invertible sheaves iff they are *linearly equivalent*, i.e., $D_1 = D_2 + (f)$ for some element f of $\mathcal{R}at^\times_{\mathrm{ev}}(X)$, or more explicitly iff $f_\alpha^{(1)} = f f_\alpha^{(2)}$ for all α. If $f \in \mathcal{R}at^\times_{\mathrm{ev}}(X)$ is a global section of an invertible sheaf $\mathcal{L} = \mathcal{O}_X(D)$ then $D + (f) \geq 0$ and vice versa. The *complete linear system* $|D| = |\mathcal{O}_X(D)|$ of a divisor (or of the corresponding invertible sheaf) is the set of all effective divisors linearly equivalent to D. So we see that if $\mathcal{L} = \mathcal{O}_X(D)$ then

$$|D| \simeq H^0(X, \mathcal{L})^\times_{\mathrm{ev}} / H^0(X, \mathcal{O}_X)^\times_{\mathrm{ev}}.$$

In case the cohomology of $\mathcal{L}$ is free of rank $p+1 \mid q$ and $H^0(X, \mathcal{O}_X)$ is just the constants $\Lambda \mid 0$, the complete linear system $|D|$ is (the set of Λ-points of) a super projective space $\mathbb{P}^{p|q}_\Lambda$. In particular, if $(X, \mathcal{O}_X)$ is a generic SKP curve and the degree d of $\mathcal{L}$ is $\geq 2g-1$ the first cohomology of $\mathcal{L}$ vanishes, the zeroth cohomology is free of rank $d+1-g \mid d+1-g$ and $|D| \simeq \mathbb{P}^{d-g|d+1-g}_\Lambda$.

Let $\hat{X} = (X, \hat{\mathcal{O}}_X)$ be the dual curve and denote by $\hat{X}^{(d)}$ the d-fold symmetric product of $\hat{X}$, see [DHS93]. This smooth supermanifold of dimension $(d \mid d)$ parametrizes effective divisors of degree d on $(X, \mathcal{O}_X)$. We have a map (called *Abelian sum*) $A : \hat{X}^{(d)} \to \mathrm{Pic}^d(X)$ sending an effective divisor D to the corresponding invertible sheaf $\mathcal{O}_X(D)$. An invertible sheaf $\mathcal{L}$ is in the image of A iff $\mathcal{L}$ has a even invertible global section: if $D \in \hat{X}^{(d)}$ and $\mathcal{L} = A(D)$ then the fiber of A at $\mathcal{L}$ is the complete linear system $|D|$. If the degree d of $\mathcal{L}$ is at least $2g-1$ $H^1(X, \mathcal{L})$ is zero and hence the cohomology of $\mathcal{L}$ is free. So in that case A is surjective and the fibers of A are all projective spaces $\mathbb{P}^{d-g|d+1-g}$ and A is in fact a fibration.

The symmetric product $\hat{X}^{(d)}$ is a universal parameter space for effective divisors of degree d. This is studied in detail by Domínguez Pérez et al. [DHS93]; we will summarize some of their results and refer to their paper for more details. (In fact they consider curves over a field, but the theory is not significantly different for curves over Λ.) A *family of effective divisors* of degree d on X parametrized by a super scheme S is a pair (S, D_S), where D_S is a Cartier divisor on $X \times_\Lambda S$ such that for any morphism $\phi : T \to S$ the induced map $(1 \times \phi)^* \mathcal{O}_{X \times S}(-D_S) \to (1 \times \phi)^* \mathcal{O}_{X \times S}$ is injective and such that for any

$s \in S$ the restriction of D_S to $X \times \{s\} \simeq X$ is an effective divisor of degree d. For example, in $X \times \hat{X}^{(d)}$ there is a canonical divisor $\Delta^{(d)}$ such if p_D is any Λ-point of $\hat{X}^{(d)}$ corresponding to a divisor D then the restriction of $\Delta^{(d)}$ to $X \times \{p_D\} \simeq X$ is just D. Then $(\hat{X}^{(d)}, \Delta^{(d)})$ is universal in the sense that for any family (S, D_S) there is a unique morphism $\Psi : S \to \hat{X}^{(d)}$ such that $D_S = \Psi^* \Delta^{(d)}$.

A *family of invertible sheaves* of degree d on X parametrized by a super scheme S is a pair $(S, \mathcal{L}_S)$, where $\mathcal{L}_S$ is an invertible sheaf on $X \times_\Lambda S$ such that for any $s \in S$ the restriction of $\mathcal{L}_S$ to $X \times \{s\}$ is a sheaf of degree d on X. For example, $(\hat{X}^{(d)}, \mathcal{O}_{X \times \hat{X}^{(d)}}(\Delta^{(d)})$ is a family of invertible sheaves of degree d. Two families $(S, \mathcal{L}_1)$, $(S, \mathcal{L}_2)$ are equivalent if $\mathcal{L}_1 = \mathcal{L}_2 \otimes \pi_S^* \mathcal{N}$, where $\pi_S : X \times S \to S$ is the canonical projection and $\mathcal{N}$ is an invertible sheaf on S. For example, fix a point x of X; then$(\hat{X}^{(d)}, \mathcal{O}_{X \times \hat{X}^{(d)}}(\Delta^{(d)}))$ is equivalent to $(\hat{X}^{(d)}, \mathcal{R}_x))$, where $\mathcal{R}_x = \mathcal{O}_{X \times \hat{X}^{(d)}}(\Delta^{(d)}) \otimes \pi^*_{\hat{X}^{(d)}}[\mathcal{O}_{X \times \hat{X}^{(d)}}(\Delta^{(d)})|_{\{x\} \times \hat{X}^{(d)}})]^{-1}$. The family $(\hat{X}^{(d)}, \mathcal{R}_x)$ is normalized: it has the property that $\mathcal{R}_x$ restricted to $\{x\} \times \hat{X}^{(d)}$ is canonically trivial. Now consider the mapping $(1 \times A) : X \times \hat{X}^{(d)} \to X \times \mathrm{Pic}^d(X)$ and the direct image $\mathcal{P}_x^{(d)} = (1 \times A)_* \mathcal{R}_x$.

Theorem 2.13.1. *Let $(X, \mathcal{O}_X)$ be a generic SKP curve. Let $d \geq 2g - 1$. Then $\mathcal{P}_x^{(d)}$ is a Poincaré sheaf on $X \times \mathrm{Pic}^d(X)$, i.e., $(\mathrm{Pic}^d(X), \mathcal{P}_x^{(d)})$ is a family of invertible sheaves of degree d that is universal in the sense that for any family $(S, \mathcal{L})$ of degree d invertible sheaves there is a unique morphism $\phi : S \to \mathrm{Pic}^d(X)$ so that $\mathcal{L} = \phi^* \mathcal{P}_x^{(d)}$. Furthermore $\mathcal{P}_x^{(d)}$ is normalized so that the restriction to $\{x\} \times \mathrm{Pic}^d(X)$ is canonically trivial.*

2.14. **Berezinian bundles.** We continue with the study of a generic SKP curve $(X, \mathcal{O}_X)$; we fix an integer n and write $\mathcal{P}$ for $\mathcal{P}_X^n$, the Poincaré sheaf on $X \times \mathrm{Pic}^n(X)$. Let $\mathcal{L}_s$ be an invertible sheaf corresponding to $s \in \mathrm{Pic}^n(X)$. The cohomology groups $H^i(X, \mathcal{L}_s)$ will vary as s varies over $\mathrm{Pic}^n(X)$ and can in general be nonfree, as we have seen. Even if the cohomology groups are free Λ-modules their ranks will jump. Still it is possible to define an invertible sheaf Ber over $\mathrm{Pic}^n(X)$ with fiber at s the line

$$\mathrm{ber}(H^0(X, \mathcal{L}_s)) \otimes \mathrm{ber}^*(H^1(X, \mathcal{L}_s)),$$

in case $\mathcal{L}_s$ has free cohomology. Here $\mathrm{ber}(M)$ for a free rank $d \mid \delta$ Λ-module with basis $\{f_1, \ldots, f_d, \phi_1, \ldots, \phi_\delta\}$ is the rank 1 Λ-module with generator $B[f_1, \ldots, f_d, \phi_1, \ldots, \phi_\delta]$. If we are given another basis $\{f'_1, \ldots, f'_d, \phi'_1, \ldots, \phi'_\delta\} = g \cdot \{f_1, \ldots, f_d, \phi_1, \ldots, \phi_\delta\}$, with $g \in Gl(d \mid$

$\delta, \Lambda)$, we have the relation

$$B[f'_1, \dots, f'_d, \phi'_1, \dots, \phi'_\delta] = \mathrm{ber}(g) B[f_1, \dots, f_d, \phi_1, \dots, \phi_\delta].$$

Similarly $\mathrm{ber}^*(M)$ is defined using the inverse homomorphism ber^*. Here ber and ber^* are the group homomorphisms defined in (C.3).

The invertible sheaf $\mathcal{L}_s$ is obtained from the Poincaré sheaf via $i_s^*\mathcal{P}$. We can reformulate this somewhat differently: $\mathcal{P}$ is an $\mathcal{O}_{\mathrm{Pic}^n(X)}$-module and for every Λ-point s of $\mathrm{Pic}^n(X)$, via the homomorphism $s^\sharp : \mathcal{O}_{\mathrm{Pic}^n(X)} \to \Lambda$, also Λ becomes an $\mathcal{O}_{\mathrm{Pic}^n(X)}$-module, denoted by Λ_s. Then $\mathcal{L}_s = i_s^*\mathcal{P} = \mathcal{P} \otimes_{\mathcal{O}_{\mathrm{Pic}^n(X)}} \Lambda_s$. It was Grothendieck's idea to study the cohomology of $\mathcal{P} \otimes M$ for arbitrary $\mathcal{O}_{\mathrm{Pic}^n(X)}$-modules M. We refer to Kempf ([Kem83]) for an excellent discussion and more details on these matters.

The basic fact is that, given the Poincaré bundle $\mathcal{P}$ on $X \times \mathrm{Pic}^n(X)$, there is a homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ of locally free coherent sheaves on $\mathrm{Pic}^n(X)$ such that we get for any sheaf of $\mathcal{O}_{\mathrm{Pic}^n(X)}$-modules M an exact sequence

$$0 \to H^0(X \times \mathrm{Pic}^n(X), \mathcal{P} \otimes M) \to \mathcal{F} \otimes M \overset{\alpha \times 1_M}{\to} \mathcal{G} \otimes M \to$$
$$\to H^1(X \times \mathrm{Pic}^n(X), \mathcal{P} \otimes M) \to 0.$$

The proof of this is the same as for the analogous statement in the classical case, see [Kem83].

Now $\mathcal{F}$ and $\mathcal{G}$ are locally free, so for small enough open sets U on $\mathrm{Pic}^n(X)$ one can define $\mathrm{ber}(\mathcal{F}(U))$ and $\mathrm{ber}^*(\mathcal{G}(U))$. This globalizes to invertible sheaves $\mathrm{ber}(\mathcal{F})$ and $\mathrm{ber}^*(\mathcal{G})$. Next we form the "Berezinian of the cohomology of $\mathcal{P}$" by defining $\mathrm{Ber} = \mathrm{ber}(\mathcal{F}) \otimes \mathrm{ber}^*(\mathcal{G})$. Finally one proves, as in Soulé, [Sou92], VI.2, Lemma 1, that Ber does not depend, up to isomorphism, on the choice of homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$.

Theorem 2.14.1. *The first Chern class of the* Ber *bundle is zero.*

We will prove this theorem in subsection 4.2, after the introduction of the infinite super Grassmannian and the Krichever map. The topological triviality of the Ber bundle is a fundamental difference from the situation of classical curves: there the determinant bundle on Pic is ample.

Next we consider the special case of $n = g - 1$. In this case, because of Riemann-Roch (2.22) $\mathcal{F}$ and $\mathcal{G}$ have the same rank. Indeed, locally $\alpha(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is given, after choosing bases, by a matrix over $\mathcal{O}_{\mathrm{Pic}^n(X)}(U)$ of size $d \mid \delta \times e \mid \epsilon$, say. If we fix a Λ-point s in U we get a homomorphism $\alpha(U)_s : \mathcal{F}(U) \otimes \Lambda_s \to \mathcal{G}(U) \otimes \Lambda_s$ represented by a matrix over Λ. The kernel and cokernel are the cohomology groups of $\mathcal{L}_s$ and these have the same rank by Riemann-Roch. On the other hand if the kernel and cokernel of a matrix over Λ are free we have rank of kernel − rank of cokernel= $d - e \mid \delta - \epsilon = 0 \mid 0$. So $\alpha(U)$ is a square matrix. This allows us to define a map

$$\mathrm{ber}(\alpha) : \mathrm{ber}(\mathcal{F}) \to \mathrm{ber}(\mathcal{G}).$$

But this is a (non-holomorphic!) section of $\mathrm{ber}^*(\mathcal{F}) \otimes \mathrm{ber}(\mathcal{G})$, i.e., of the dual Berezinian bundle $\mathcal{P}^*$ on Pic^{g-1}, because of the non-polynomial (rational) character of the Berezinian. This section $\mathrm{ber}(\alpha)$ is essential for the definition of the τ-function in subsection 3.4.

2.15. **Bundles on the Jacobian; theta functions.** We continue with X being a generic SKP curve. Super theta functions will be defined as holomorphic sections of certain ample bundles on $J = \mathrm{Jac}(X)$, when such bundles exist. (As usual, the existence of ample invertible sheaves is necessary and sufficient for projective embeddability.) Given one such bundle, all others with the same Chern class c_1 are obtained by tensor product with bundles having trivial Chern class, so we begin by determining these, that is, computing $\mathrm{Pic}^0(J)$. As we briefly discussed in subsection 2.12 J is the quotient of the affine super space $V = \mathbb{A}^{g|g-1} = \mathrm{Spec}\,\Lambda[z_1, \dots, z_g, \eta_1, \dots, \eta_{g-1}]$ by the lattice L generated by the columns of the transposed period matrix:

$$\begin{aligned} \lambda_i : &\quad z_j \to z_j + \delta_{ij}, &\quad \eta_\alpha \to \eta_\alpha, \\ \lambda_{i+g} : &\quad z_j \to z_j + (Z_e)_{ij}, &\quad \eta_\alpha \to \eta_\alpha + (Z_o)_{i\alpha}, \quad i = 1, 2, \dots, g. \end{aligned} \tag{2.37}$$

We will often omit the parity labels e, o on Z, since the index structure makes clear which is meant.

Any line bundle $\mathcal{L}$ on such a supertorus J lifts to a trivial bundle on the covering space V. A section of $\mathcal{L}$ lifts to a function on which the translations λ_i act by multiplication by certain invertible holomorphic functions, the *multipliers* of $\mathcal{L}$. We can factor the quotient map $V \to J$ through the cylinder V/L_0, where L_0 is the subgroup of L generated by the first g λ_i only. Since holomorphic line bundles on a cylinder are trivial, this means that the multipliers for L_0 can always be taken as unity. We have $\mathrm{Pic}^0(J) \cong H^1(J, \mathcal{O}_{\mathrm{ev}})/H^1(J, \mathbb{Z})$. It is very convenient to compute the numerator as the group cohomology $H^1(L, \mathcal{O}_{\mathrm{ev}})$ of L acting on the even functions on the covering space V, in part because the cocycles for this complex are precisely (the logarithms of) the multipliers. For the basics of group cohomology, see for example [Sil86, Mum70]. In particular, factoring out the subgroup L_0 reduces our problem to computing $H^1(L/L_0, \mathcal{O}^{L_0})$, the cohomology of the quotient group acting on the L_0-invariant functions.

A 1-cochain for this complex assigns to each generator of L/L_0 an even function (log of the multiplier) invariant under each shift $z_j \to$

$z_j + 1$,

$$\lambda_{i+g} \mapsto F^i(z,\eta) = \sum_{\vec{n}} F^i_{\vec{n}}(\eta) e^{2\pi i \vec{n}\cdot\vec{z}}.$$

It is a cocycle if the multiplier induced for every sum $\lambda_{i+g} + \lambda_{j+g}$ is independent of the order of addition, which amounts to the symmetry of the matrix $\Delta_i F^j$ giving the change in F^j under the action of λ_{i+g}:

$$F^i(z_k + Z_{jk}, \eta_\alpha + Z_{j\alpha}) - F^i(z_k, \eta_\alpha) = F^j(z_k + Z_{ik}, \eta_\alpha + Z_{i\alpha}) - F^j(z_k, \eta_\alpha),$$

or, in terms of Fourier coefficients,

$$F^i_{\vec{n}}(\eta_\alpha + Z_{j\alpha}) e^{2\pi i \sum_k n_k Z_{jk}} - F^i_{\vec{n}}(\eta) = F^j_{\vec{n}}(\eta_\alpha + Z_{i\alpha}) e^{2\pi i \sum_k n_k Z_{ik}} - F^j_{\vec{n}}(\eta).$$

One does not have to allow for an integer ambiguity in the logarithms of the multipliers in these equations, precisely because we are considering bundles with vanishing Chern class. The coboundaries are of the form,

$$\lambda_{i+g} \mapsto A(z,\eta) - A(z_k + Z_{ik}, \eta_\alpha + Z_{i\alpha})$$

for a single function A, that is, those cocycles for which

$$F^i_{\vec{n}}(\eta) = A_{\vec{n}}(\eta) - A_{\vec{n}}(\eta_\alpha + Z_{i\alpha}) e^{2\pi i \sum_k n_k Z_{ik}}.$$

This equation has the form,

$$F^i_{\vec{n}}(\eta) = A_{\vec{n}}(\eta)(1 - e^{2\pi i \sum_k n_k Z_{ik}}) + O(Z_o).$$

The point now is that, by the linear independence of the columns of Z_e^{red}, for any $\vec{n} \neq \vec{0}$ there is some choice of i for which the reduced part of the exponential in the last equation differs from unity. This ensures that, for this i, the equation is solvable for $A_{\vec{n}}$, first to zeroth order in Z_o and then to all orders by iteration. Adding this coboundary to the cocycle produces one for which $F^i_{\vec{n}} = 0$, whereupon the cocycle conditions imply $F^j_{\vec{n}} = 0$ for all $\vec{n} \neq \vec{0}$ and all j as well.

Thus the only potentially nontrivial cocycles are independent of z_i. In the simplest case, when the odd period matrix $Z_o = 0$, all such cocycles are indeed nontrivial, and we have an analog of the classical fact that bundles of trivial Chern class are specified by g constant multipliers. Here a cocycle is specified by giving g even elements $F^i_{\vec{0}}(\eta)$ in the exterior algebra $\Lambda[\eta_\alpha]$ (elements of $H^0(J, \mathcal{O}_J)$), leading to $\dim \mathrm{Pic}^0(J) = g^{2^{g-2}} \mid g^{2^{g-2}}$ (the number of η_α is $g-1$). In general, when $Z_o \neq 0$, not all cochains specified in this way will be cocycles, and some cocycles will be trivial: $\mathrm{Pic}^0(J)$ will be smaller, and in general not a supermanifold.

As to the existence of ample line bundles, let us examine in the super case the classical arguments leading to the necessary and sufficient Riemann conditions [GH78, LB92]. The Chern class of a very ample bundle is represented in de Rham cohomology by a $(1,1)$ form obtained as the pullback of the Chern class of the hyperplane bundle via a projective embedding. We can introduce real even coordinates $x_i, i = 1, \ldots, 2g$ for J dual to the basis λ_i of the lattice L, meaning that $x_j \to x_j + \delta_{ij}$ under the action of λ_i. The associated real odd coordinates $\xi_\alpha, \alpha = 1, \ldots, 2g-2$ can be taken to be globally defined because every real supermanifold is split. The relation between the real and complex coordinates can be taken to be

$$\begin{aligned} z_j &= x_j + \sum_{i=1}^{g} Z_{ij} x_{i+g}, \ j = 1, \ldots, g, \\ \eta_\alpha &= \xi_\alpha + i\xi_{\alpha+g-1} + \sum_{i=1}^{g} Z_{i\alpha} x_{i+g}, \ \alpha = 1, \ldots, g-1. \end{aligned}$$

The de Rham cohomology is isomorphic to that of the reduced torus and can be represented by translation-invariant forms in the dx_i. The Chern class represented by a form $\sum_{i=1}^{g} \delta_i \, dx_i \, dx_{i+g}$ is called a polarization of type $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_g)$ with elementary divisors the positive integers δ_i. We consider principal polarizations $\delta_i = 1$ only, because nontrivial nonprincipal polarizations generically do not exist, even on the reduced torus [Lef28]. Furthermore, a nonprincipal polarization is always obtained by pullback of a principal one from another supertorus whose lattice L' contains L as a sublattice of finite index [GH78]. Reexpressing the Chern form in complex coordinates, the standard calculations lead to the usual Riemann condition $Z_e = Z_e^t$ to obtain a $(1,1)$ form. Together with the positivity of the imaginary part of the reduced matrix, the symmetry of Z_e (in some basis) is necessary and sufficient for the existence of a $(1,1)$ form with constant coefficients representing the Chern class. This can be viewed as the cocycle condition, symmetry of $\Delta_i F^j$, for the usual multipliers of a theta bundle, $F^j = -2\pi i z_j$.

The usual argument that the $(1,1)$ form representing the Chern class can always be taken to have constant coefficients depends on Hodge theory, particularly the Hodge decomposition of cohomology, for a Kähler manifold such as a torus. This does not hold in general for a supertorus with $Z_o \neq 0$. For example, $H^1_{\mathrm{dR}}(J)$ is generated by the $2g$ 1-forms dx_i, whereas $H^{1,0}(J)$ contains the $g \mid g-1$ nontrivial forms dz_i, $d\eta_\alpha$, with certain nilpotent multiples of the latter being trivial. Indeed, since by (2.37) η_α is defined modulo entries of column α of Z_o, $\epsilon\eta_\alpha$ is a global function and $\epsilon d\eta_\alpha$ is exact when $\epsilon \in \Lambda$ annihilates these entries. Thus, $H^{1,0}(J)$ cannot be a direct summand in $H^1_{\mathrm{dR}}(J)$. Correspondingly, some η-dependent multipliers $F^j = -2\pi i z_j + \cdots$ may satisfy the cocycle condition and give ample line bundles. We do not know a simple necessary condition for a Jacobian to admit such polarizations.

When Z_e is symmetric, we can construct theta functions explicitly. Consider first the trivial case with $Z_o = 0$ as well. Then the standard Riemann theta function $\Theta(z; Z_e)$ gives a super theta function on $\mathrm{Jac}(X)$, where $\Theta(z; Z_e)$ is defined by Taylor expansion in the nilpotent part of Z_e as usual. It has of course the usual multipliers,

(2.38)
$$\Theta(z_j + \delta_{ij}; Z_e) = \Theta(z_j; Z_e), \quad \Theta(z_j + Z_{ij}; Z_e) = e^{-\pi i (2z_i + Z_{ii})}\Theta(z_j; Z_e).$$

Multiplication of $\Theta(z; Z_e)$ by any monomial in the odd coordinates η_α gives another, even or odd, theta function having the same multipliers, whereas translation of the argument z by polynomials in the η_α leads to the multipliers for another bundle with the same Chern class.

In the general case with $Z_o \neq 0$, theta functions with the standard multipliers can be constructed as follows. Such functions must obey

$$H(z_j + \delta_{ij}, \eta_\alpha; Z) = H(z_j, \eta_\alpha; Z),$$
$$H(z_j + Z_{ij}, \eta_\alpha + Z_{i\alpha}; Z) = e^{-\pi i (2z_i + Z_{ii})} H(z_j, \eta_\alpha; Z).$$

The function $\Theta(z; Z_e)$ is a trivial example independent of η; to obtain others one checks that when H satisfies these relations then so does

$$H_\alpha = \left(\eta_\alpha + \frac{1}{2\pi i} \sum_k Z_{k\alpha} \frac{\partial}{\partial z_k} \right) H.$$

Applying this operator repeatedly one constructs super theta functions $\Theta_{\alpha\cdots\gamma}$ reducing to $\eta_\alpha \cdots \eta_\gamma \Theta(z; Z_e)$ when $Z_o = 0$.

"Translated" theta functions which are sections of other bundles having the same Chern class can be obtained by literally translating the arguments of these only in the simplest cases. Constant shifts in the multiplier exponents F^j can be achieved by constant shifts of the arguments z_j. Shifts linear in the η_α are obtained by $z_j \to z_j + \eta_\alpha \Gamma_{\alpha j}$, which is a change in the chosen basis of holomorphic differentials on X, see the discussion after (2.24). The resulting theta functions have the new period matrix $Z_e + Z_o \Gamma$. More generally, translated theta functions can be obtained by the usual method of determining their Fourier coefficients from the recursion relations following from the desired multipliers. We do not know an explicit expression for them in terms of conventional theta functions.

It is easy to see that any meromorphic function F on the Jacobian can be rationally expressed in terms of the theta functions we have defined. Expand $F(z,\eta) = \sum_{IJ} \beta_I \eta_J F_{IJ}(z)$ in the generators of $\Lambda[\eta_\alpha]$, with multi-indices I, J. Then the zeroth-order term F_{00} is a meromorphic function on the reduced Jacobian, hence a rational expression in ordinary theta functions. Using Z_e as the period matrix argument of these theta functions gives a meromorphic function on the Jacobian itself, whose reduction agrees with F_{00}. Subtract this expression from F to get a meromorphic function on the Jacobian whose zeroth-order term vanishes, and continue inductively, first in J, then in I. For example, $F_{0\alpha}$ is equal, to lowest order in the β's, to a rational expression in theta functions of which one numerator factor is a Θ_α. Subtracting this expression removes the corresponding term in F while only modifying other terms of higher order in β's.

3. Super Grassmannian, τ-function and Baker function.

3.1. Super Grassmannians. In this subsection we will introduce an infinite super Grassmannian and related constructions. The infinite Grassmannian of Sato ([Sat85]) or of Segal-Wilson ([SW85]) consists (essentially) of "half infinite dimensional" vector subspaces W of an infinite dimensional vector space H such that the projection on a fixed subspace H_- has finite dimensional kernel and cokernel. In the super category we replace this by the super Grassmannian of free "half infinite rank" Λ-modules of an infinite rank free Λ-module H such that the kernel and cokernel of the projection on H_- are a submodule respectively a quotient module of a free finite rank Λ-module. In [Sch89] a similar construction can be found, but it seems that there $\Lambda = \mathbb{C}$ is taken as is also the case in [Mul90]. This is too restrictive for our purposes involving algebraic super curves over nonreduced base ring Λ.

Let $\Lambda^{\infty|\infty}$ be the free Λ-module $\Lambda[z, z^{-1}, \theta]$ with z an even and θ an odd variable. Introduce the notation

$$e_i = z^i, \quad e_{i-\frac{1}{2}} = z^i\theta, \quad i \in \mathbb{Z}. \tag{3.1}$$

We will think of an element $h = \sum_{i=-N}^{\infty} h_i e_i$, $h_i \in \Lambda$ of $\Lambda^{\infty|\infty}$ not only as a series in z, θ but also as an infinite column vector:

$$h = (\dots, 0, \dots, h_{-1}, h_{-\frac{1}{2}}, h_0, h_{\frac{1}{2}}, h_1, \dots, 0, \dots)^t$$

Introduce on $\Lambda^{\infty|\infty}$ an odd Hermitian product

$$\langle f(z,\theta), g(z,\theta)\rangle = \frac{1}{2\pi i}\oint \frac{dz}{z} d\theta \overline{f(z,\theta)} g(z,\theta) = \tag{3.2}$$
$$= \frac{1}{2\pi i}\oint (\overline{f_{\bar 0}} g_{\bar 1} + \overline{f_{\bar 1}} g_{\bar 0})\frac{dz}{z},$$

where $\overline{f(z,\theta)}$ is the extension of the complex conjugation of Λ (see Appendix B) to $\Lambda^{\infty|\infty}$ by $\overline{z} = z^{-1}$ and $\overline{\theta} = \theta$, and $f(z,\theta) = f_{\bar 0} + \theta f_{\bar 1}$, and similarly for g. Let H be the completion of $\Lambda^{\infty|\infty}$ with respect to the Hermitian inner product.

We have a decomposition $H = H_- \oplus H_+$, where H_- is the closure of the subspace spanned by e_i for $i \leq 0$, and H_+ is the closure of the space spanned by e_i with $i > 0$, for $i \in \frac{1}{2}\mathbb{Z}$.

The super Grassmannian $\mathcal{S}\mathrm{gr}$ is the collection of all free closed Λ-modules $W \subset H$ such that the projection $\pi_- : W \to H_-$ is super Fredholm, i.e., the kernel and cokernel are a submodule respectively a quotient module of a free finite rank Λ-module.

Example 3.1.1. Let W be the closure of the subspace generated by $\delta + z, \theta$ and $z^i, z^i\theta$ for $i \leq -1$, for δ a nilpotent even constant. Let $A \subset \Lambda$ be the ideal of annihilators of δ. Then W is free and the kernel of π_- is $A(\delta+z) \subset \Lambda(\delta+z)$ and the cokernel is isomorphic to $\Lambda/\Lambda\delta$. □

Let I be the subset $\{i \in \frac{1}{2}\mathbb{Z} \mid i \leq 0\}$. We consider matrices with coefficients in Λ of size $\frac{1}{2}\mathbb{Z} \times I$:

$$\mathcal{W} = (W_{ij}) \quad \text{where } i \in \frac{1}{2}\mathbb{Z},\ j \in I.$$

An even matrix of this type is called an *admissible frame* for $W \in \mathcal{S}\mathrm{gr}$ if the closure of the subspace spanned by the columns of $\mathcal{W}$ is W and if moreover in the decomposition $\mathcal{W} = \begin{pmatrix} W_- \\ W_+ \end{pmatrix}$ induced by $H = H_- \oplus H_+$ the operator $W_- : H_- \to H_-$ differs from the identity by an operator of super trace class and $W_+ : H_- \to H_+$ is compact.

Let $Gl(H_-)$ be the group of invertible maps $1+X : H_- \to H_-$ with X super trace class. Then the super frame bundle $\mathcal{S}\mathrm{fr}$, the collection of all pairs $(W, \mathcal{W})$ with $\mathcal{W}$ an admissible frame for $W \in \mathcal{S}\mathrm{gr}$, is a principal $Gl(H_-)$ bundle over the super Grassmannian. Elements of $Gl(H_-)$ have a well defined berezinian, see [Sch89] for some details. This allows us to define two associated line bundles $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ and $\mathrm{Ber}^*(\mathcal{S}\mathrm{gr})$ on $\mathcal{S}\mathrm{gr}$. More explicitly, an element of $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ is an equivalence class of triples $(W, \mathcal{W}, \lambda)$, with $\mathcal{W}$ a frame for W, $\lambda \in \Lambda$; here $(W, \mathcal{W}g, \lambda)$ and $(W, \mathcal{W}, \mathrm{ber}(g)\lambda)$ are equivalent for $g \in Gl(H_-)$. For $\mathrm{Ber}^*(\mathcal{S}\mathrm{gr})$ replace

$\mathrm{ber}(g)$ by $\mathrm{ber}^*(g)$. For simplicity we shall write $(\mathcal{W}, \lambda)$ for $(W, \mathcal{W}, \lambda)$, as $\mathcal{W}$ determines W uniquely.

The two bundles $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ and $\mathrm{Ber}^*(\mathcal{S}\mathrm{gr})$ each have a canonical section. Let $\mathcal{W}$ be a frame for $W \in \mathcal{S}\mathrm{gr}$ and write $\mathcal{W} = \begin{pmatrix} W_- \\ W_+ \end{pmatrix}$ as above. Then

$$\sigma(W) = (\mathcal{W}, \mathrm{ber}(W_-)), \quad \sigma^*(W) = (\mathcal{W}, \mathrm{ber}^*(W_-)), \tag{3.3}$$

are sections of $\mathrm{Ber}^*(\mathcal{S}\mathrm{gr})$ and $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$, respectively. It is a regrettable fact of life that neither of these sections is holomorphic; indeed there are no global sections to $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ or $\mathrm{Ber}^*(\mathcal{S}\mathrm{gr})$ at all, see [Man88]. This is a major difference between classical geometry and super geometry.

3.2. **The Chern class of $\mathrm{Ber}(\mathcal{S}\mathbf{gr})$ and the $gl_{\infty|\infty}$ cocycle.** First we summarize some facts about complex supermanifolds that are entirely analogous to similar facts for ordinary complex manifolds. Then we apply this to the super Grassmannian, following the treatment in [PS86] of the classical case.

Let M be a complex supermanifold. The Chern class of an invertible sheaf $\mathcal{L}$ on M is an element $c_1(\mathcal{L}) \in H^2(M, \mathbb{Z})$. By the sheaf inclusion $\mathbb{Z} \to \Lambda$ and the de Rham theorem $H^2(M, \Lambda) \simeq H^2_{\mathrm{dR}}(M)$ we can represent $c_1(\mathcal{L})$ by a closed two form on M. On the other hand, if $\nabla : \mathcal{L} \to \mathcal{L} \otimes \mathcal{A}^1$, with $\mathcal{A}^1$ the sheaf of smooth 1-forms, is a connection compatible with the complex structure, the curvature F of ∇ is also a two form. By the usual proof (see e.g., [GH78]) we find that $c_1(\mathcal{L})$ and F are equal, up to a factor of $i/2\pi$.

We can locally calculate the curvature on an invertible sheaf $\mathcal{L}$ by introducing a Hermitian metric $\langle\, , \rangle$ on it: if $s, t \in \mathcal{L}(U)$ then $\langle s, t\rangle(m)$ is a smooth function in $m \in U$ taking values in Λ, linear in t and satisfying $\langle s, t\rangle(m) = \overline{\langle t, s\rangle(m)}$. Choose a local generator e of $\mathcal{L}$ and let $h = \langle e, e\rangle$. The curvature is then $F = \bar{\partial}\partial \log h$, with $\partial = \sum dz_i \frac{d}{dz_i} + \sum d\theta_\alpha \frac{\partial}{\partial \theta_\alpha}$ and $\bar{\partial}$ defined by a similar formula.

Now consider the invertible sheaf $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ on $\mathcal{S}\mathrm{gr}$. If $s = (\mathcal{W}, \lambda)$ is a section the square length is defined to be $\langle s, s\rangle = \bar{\lambda}\lambda \,\mathrm{ber}(\mathcal{W}^H \mathcal{W})$, where superscript ^{H} indicates conjugate transpose. Of course, this metric is not defined everywhere on $\mathcal{S}\mathrm{gr}$ because of the rational character of ber, but we are interested in a neighborhood of the point W_0 with standard frame $\mathcal{W}_0 = \begin{pmatrix} 1_{H_-} \\ 0 \end{pmatrix}$ where there is no problem. The tangent space at W_0 can be identified with the space of maps $H_- \to H_+$, or, more concretely, by matrices with the columns indexed by $I = \{i \in \frac{1}{2}\mathbb{Z} \mid i \leq 0\}$ and with rows indexed by the complement of I. Let x, y be two tangent vectors at W_0. Then the curvature at W_0 is calculated to be

$$F(x,y) = \bar{\partial}\partial \log h(x,y) = Str(x^H y - y^H x), \tag{3.4}$$

where we take as local generator $e = \sigma$, the section defined by (3.3), so that $h = \langle \sigma, \sigma \rangle$. We can map the tangent space at W_0 to the Lie super algebra $gl_{\infty|\infty}(\Lambda)$ via $x \mapsto \begin{pmatrix} 0 & -x^H \\ x & 0 \end{pmatrix}$. Here $gl_{\infty|\infty}(\Lambda)$ is the Lie super algebra corresponding to the Lie super group $Gl_{\infty|\infty}(\Lambda)$ of infinite even invertible matrices g (indexed by $\frac{1}{2}\mathbb{Z}$) with block decompostion $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with b, c compact and a, d super Fredholm. We see that (3.4) is the pullback under this map of the cocycle on $gl_{\infty|\infty}(\Lambda)$ (see also [KvdL87]) given by

$$\begin{aligned} c : gl_{\infty|\infty}(\Lambda) \times gl_{\infty|\infty}(\Lambda) &\to \Lambda \\ (X, Y) &\mapsto \frac{1}{4} \operatorname{Str}(J[J,X][J,Y]), \end{aligned} \tag{3.5}$$

where $J = \begin{pmatrix} 1_{H_-} & 0 \\ 0 & -1_{H_+} \end{pmatrix}$. In terms of the block decomposition of X, Y we have

$$c(X,Y) = \operatorname{Str}(c_X b_Y - b_X c_Y).$$

The natural action of $Gl_{\infty|\infty}(\Lambda)$ on $\mathcal{S}\mathrm{gr}$ lifts to a projective action on $\operatorname{Ber}(\mathcal{S}\mathrm{gr})$; the cocycle c describes infinitesimally the obstruction for this projective action to be a real action. Indeed, if $g_1 = \exp(f_1), g_2 = \exp(f_2)$ and $g_3 = g_1 g_2$ are all in the open set of $Gl_{\infty|\infty}(\Lambda)$ where the $_{--}$ blocks a_i are invertible, the action on a point of $\operatorname{Ber}(\mathcal{S}\mathrm{gr})$ is given by

$$g_i \circ (\mathcal{W}, \lambda) = (g_i \mathcal{W} a_i^{-1}, \lambda). \tag{3.6}$$

(One checks as in [SW85] that if $\mathcal{W}$ is an admissible basis then so is $g\mathcal{W} g_{--}^{-1}$.) Then we have

$$g_1 \circ g_2 \circ (\mathcal{W}, \lambda) = \exp[c(f_1, f_2)] g_3 \circ (\mathcal{W}, \lambda).$$

We can also introduce the *projective multiplier* $C(g_1, g_2)$ for elements g_1 and g_2 that commute in $Gl_{\infty|\infty}(\Lambda)$:

$$g_1 \circ g_2 \circ g_1^{-1} \circ g_2^{-1} (\mathcal{W}, \lambda) = C(g_1, g_2)(\mathcal{W}, \lambda), \tag{3.7}$$

where $C(g_1, g_2) = \exp[S(f_1, f_2)]$ if $g_i = \exp(f_i)$ and

$$S(f_1, f_2) = \operatorname{Str}([f_1, f_2]). \tag{3.8}$$

We will in subsection 4.2 use the projective multiplier to show that the Chern class of the Berezinian bundle on $\operatorname{Pic}^0(X)$ is trivial.

3.3. **The Jacobian super Heisenberg algebra.** In the theory of the KP hierarchy an important role is played by a certain Abelian subalgebra of the infinite matrix algebra and its universal central extension, loosely referred to as the (principal) Heisenberg subalgebra. In this subsection we introduce one of the possible analogs of this algebra in the super case.

Let the *Jacobian super Heisenberg algebra* be the Λ-algebra $\mathcal{J}\mathrm{Heis} = \Lambda[z, z^{-1}, \theta]$. Of course, this is as a Λ-module the same as $\Lambda^{\infty|\infty}$ but now we allow multiplication of elements. When convenient we will identify the two; in particular we will use the basis $\{e_i\}$ of (3.1) also for $\mathcal{J}\mathrm{Heis}$. We think of elements of $\mathcal{J}\mathrm{Heis}$ as infinite matrices in $gl_{\infty|\infty}(\Lambda)$: if E_{ij} is the elementary matrix with all entries zero except for the ijth entry which is 1, then

$$e_i = \sum_{n\in\mathbb{Z}} E_{n+i,n} + E_{n+i-\frac{1}{2},n-\frac{1}{2}}, \quad e_{i-\frac{1}{2}} = \sum_{n\in\mathbb{Z}} E_{n+i-\frac{1}{2},n}.$$

We have a decomposition $\mathcal{J}\mathrm{Heis} = \mathcal{J}\mathrm{Heis}_- \oplus \mathcal{J}\mathrm{Heis}_+$ in subalgebras $\mathcal{J}\mathrm{Heis}_- = z^{-1}\Lambda[z^{-1}, \theta]$ and $\mathcal{J}\mathrm{Heis}_+ = \Lambda[z, \theta]$. Elements of $\mathcal{J}\mathrm{Heis}_+$ correspond to lower triangular matrices and elements of $\mathcal{J}\mathrm{Heis}_-$ to upper triangular ones. By exponentiation we obtain from $\mathcal{J}\mathrm{Heis}_-$ and $\mathcal{J}\mathrm{Heis}_+$ two subgroups G_- and G_+ of $Gl_{\infty|\infty}(\Lambda)$, generated by

$$g_\pm(t) = \exp(\sum_{i\in\pm I} t_i e_i),$$

where $t_i \in \Lambda$ is homogeneous of the same parity as e_i (and t_i is zero for almost all i, say).

For an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of G_+ the block b vanishes, whereas if $g \in G_-$ the block $c = 0$. In either case the diagonal block a is invertible and we can lift the action of either G_- or G_+ to a (potentially projective) action on $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ and $\mathrm{Ber}^*(\mathcal{S}\mathrm{gr})$, via (3.6). Since the cocycle (3.5) is zero when restricted to both $\mathcal{J}\mathrm{Heis}_-$ and $\mathcal{J}\mathrm{Heis}_+$ we get an honest action of the Abelian groups $G_\pm$ on $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ and $\mathrm{Ber}^*(\mathcal{S}\mathrm{gr})$, just as in the classical case.

In contrast with the classical case, however, as was pointed out in [Sch89], the actions of G_- and G_+ on the line bundles $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ and $\mathrm{Ber}^*(\mathcal{S}\mathrm{gr})$ mutually commute. This follows from the following Lemma.

Lemma 3.3.1. *Let $g_\pm \in G_\pm$ and write $a_\pm = \exp(f_\pm)$, with $f_\pm \in gl(H_-)$. Then*

$$\mathrm{Str}_{H_-}([f_-, f_+]) = 0,$$

so that the actions of g_- and g_+ on $\mathrm{Ber}(\mathcal{S}gr)$ and $\mathrm{Ber}^*(\mathcal{S}gr)$ commute.

Proof. The elements $f_\pm$ act on H_- by multiplication by an element of $\mathcal{J}\mathrm{Heis}_\pm$, followed by projection on H_- if necessary. So write $f_\pm = \pi_{H_-} \circ \sum_{i>0} c_i^\pm z^{\pm i} + \gamma_i^\pm z^{\pm i}\theta$. To find the supertrace we need to calculate the projection on the rank $1 \mid 0$ and $0 \mid 1$ submodules of H_- generated by z^{-i} and $z^{-i}\theta$:

$$
\begin{aligned}
f_+f_-z^{-k}|_{\Lambda z^{-k}} &= f_+(\sum_{i>0} c_i^- z^{-i-k})|_{\Lambda z^{-k}} = (\sum_{i>0} c_i^+ c_i^-)z^{-k},\\
f_+f_-z^{-k}\theta|_{\Lambda z^{-k}\theta} &= (\sum_{i>0} c_i^+ c_i^-)z^{-k}\theta,\\
f_-f_+z^{-k}|_{\Lambda z^{-k}} &= f_-(\sum_{i=1}^{k} a_i^+ z^{i-k})|_{\Lambda z^{-k}} = (\sum_{i=1}^{k} c_i^+ c_i^-)z^{-k},\\
f_-f_+z^{-k}\theta|_{\Lambda z^{-k}\theta} &= f_-(\sum_{i=1}^{k} c_i^+ z^{i-k}\theta)|_{\Lambda z^{-k}\theta} = (\sum_{i=1}^{k} c_i^+ c_i^-)z^{-k}\theta.
\end{aligned}
$$

Since the super trace is the difference of the traces of the restrictions to the even and odd submodules we see that $\mathrm{Str}([f_+, f_-]) = 0$ so that, by (3.7,3.8), the actions of $G_\pm$ on $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ commute. □

3.4. **Baker functions, the full super Heisenberg algebra, and τ-functions.** We define $W \in \mathcal{S}\mathrm{gr}$ to be *in the big cell* if it has an admissible frame $\mathcal{W}^{(0)}$ of the form

$$
\mathcal{W}^{(0)} = \begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots \\
\dots & 1 & 0 & 0 & 0 \\
\dots & 0 & 1 & 0 & 0 \\
\dots & 0 & 0 & 1 & 0 \\
\dots & 0 & 0 & 0 & 1 \\
*** & * & * & * & *
\end{pmatrix},
$$

i.e. $(\mathcal{W}^{(0)})_-$ is the identity matrix. Note that the canonical sections σ and σ^* do not vanish, nor blow up, at a point in the big cell.

If $\mathcal{W}$ is any frame of a point W in the big cell we can calculate the standard frame $\mathcal{W}^{(0)}$ through quotients of Berezinians of minors of $\mathcal{W}$. Indeed, if we put $A = \mathcal{W}_-$ then the maximal minor A of $\mathcal{W}$ is invertible and we have

$$
\mathcal{W}^{(0)} A = \mathcal{W}. \tag{3.9}
$$

Write $\mathcal{W}^{(0)} = \sum w_{ij}^{(0)} E_{ij}$. Then we can solve (3.9) by Cramer's rule, (C.4), to find for $i > 0, j \le 0$:

$$w_{ij}^{(0)} = \begin{cases} \operatorname{ber}(A_j(r_i))/\operatorname{ber}(A) & \text{if } j \in \mathbb{Z}, \\ \operatorname{ber}^*(A_j(r_i))/\operatorname{ber}^*(A) & \text{if } j \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

Here $A_j(r_i)$ is the matrix obtained from A by replacing the jth row by r_i, the ith row of $\mathcal{W}$. In particular the even and odd "Baker vectors" of W, i.e. the zeroth and $-\frac{1}{2}$th column of $\mathcal{W}^{(0)}$, are given by

$$(3.10) \qquad \begin{aligned} w_{\bar 0} &= e_0 + \sum_{\substack{i>0 \\ i\in\frac{1}{2}\mathbb{Z}}} \frac{\operatorname{ber}(A_0(r_i))}{\operatorname{ber}(A)} e_i, \\ w_{\bar 1} &= e_{-\frac{1}{2}} + \sum_{\substack{i>0 \\ i\in\frac{1}{2}\mathbb{Z}}} \frac{\operatorname{ber}^*(A_{-\frac{1}{2}}(r_i))}{\operatorname{ber}^*(A)} e_i \end{aligned}$$

The corresponding "Baker functions" are obtained by using $e_i = z^i$, $e_{i-\frac{1}{2}} = z^i\theta$. Then (3.10) reads

$$(3.11) \qquad \begin{aligned} w_{\bar 0}(z,\theta) &= 1 + \sum_{i>0} z^i \frac{\operatorname{ber}(A_0(r_i)) + \operatorname{ber}(A_0(r_{i-\frac{1}{2}}))\theta}{\operatorname{ber}(A)}, \\ w_{\bar 1}(z,\theta) &= \theta + \sum_{i>0} z^i \frac{\operatorname{ber}^*(A_{-\frac{1}{2}}(r_i)) + \operatorname{ber}^*(A_{-\frac{1}{2}}(r_{i-\frac{1}{2}}))\theta}{\operatorname{ber}^*(A)}. \end{aligned}$$

Here and henceforth (unless otherwise noted) the summations run over (subsets of) the integers.

The full super Heisenberg algebra $\mathcal{S}$Heis is the extension $\mathcal{J}\text{Heis}[\frac{d}{d\theta}] = \Lambda[z, z^{-1}, \theta][\frac{d}{d\theta}]$

. This is, just as the Jacobian super Heisenberg algebra, a possible analog of the principal Heisenberg of the infinite matrix algebra used in the standard KP hierarchy, see [KvdL87]. $\mathcal{S}$Heis is non–Abelian and the restriction of the cocycle (3.5) to it is nontrivial, in contrast to the subalgebra $\mathcal{J}$Heis.

$\mathcal{S}$Heis acts in the obvious way on $\Lambda^{\infty|\infty}$ and we can represent it by infinite matrices from $gl_{\infty|\infty}(\Lambda)$. Introduce a basis for $\mathcal{S}$Heis by

$$\lambda(n) = z^{-n}(1 - \theta\frac{d}{d\theta}) = \sum_{k\in\mathbb{Z}} E_{k,k+n}, \quad f(n) = z^{-n}\frac{d}{d\theta} = \sum_{k\in\mathbb{Z}} E_{k,k+n-\frac{1}{2}},$$

$$\mu(n) = z^{-n}\theta\frac{d}{d\theta} = \sum_{k\in\mathbb{Z}} E_{k-\frac{1}{2},k-\frac{1}{2}+n}, \quad e(n) = z^{-n}\theta = \sum_{k\in\mathbb{Z}} E_{k-\frac{1}{2},k+n}.$$

We can rewrite the Baker functions as quotients of Berezinians, using $\mathcal{S}$Heis. To this end define the following even invertible matrices (over the ring $\Lambda[u,\phi,\frac{\partial}{\partial\phi}]$):

$$Q_{\bar 0}(u,\phi)=1+\sum_{n=1}^{\infty}u^n[\lambda(n)+f(n)\phi],$$
$$Q_{\bar 1}(u,\phi)=1+\sum_{n=1}^{\infty}u^n[\mu(n)+e(n)\frac{\partial}{\partial\phi}],$$

where u, resp ϕ, is an even, resp. odd, variable. We can let these matrices act on H and obtain in this way infinite vectors over the ring $\Lambda[u,\phi,\frac{\partial}{\partial\phi}]$. Also we can let these matrices act on an admissible frame and obtain a matrix over $\Lambda[u,\phi,\frac{\partial}{\partial\phi}]$.

Lemma 3.4.1. *Let $w_{\bar 0}(u,\phi)$ and $w_{\bar 1}(u,\phi)$ be the even and odd Baker functions of a point W in the big cell. For any frame $\mathcal{W}$ of W we have:*

$$w_{\bar 0}(u,\phi)=\frac{\operatorname{ber}\left([Q_{\bar 0}(u,\phi)\mathcal{W}]_-\right)}{\operatorname{ber}(A)},\quad w_{\bar 1}(u,\phi)=\frac{\operatorname{ber}^*\left([Q_{\bar 1}(u,\phi)\mathcal{W}]_-\right)\phi}{\operatorname{ber}^*(A)},$$

with $A=\mathcal{W}_-$.

Proof. Let r_i, $r_{i,\bar 0}$ and $r_{i,\bar 1}$, be respectively the ith row of $\mathcal{W}$, $Q_{\bar 0}(u,\phi)\mathcal{W}$ and of $Q_{\bar 1}(u,\phi)\mathcal{W}$. Then one calculates that for $i\in\mathbb{Z}$ we have $r_{i-\frac12,\bar 0}=r_{i-\frac12}$, and $r_{i,\bar 1}=r_i$ and :

$$\begin{aligned}
r_{i,\bar 0}&=r_i+\sum_{k\geq 1}u^k(r_{i+k}+r_{i+k-\frac12}\phi),\\
&=r_i+u(r_{i+1,\bar 0}+r_{i+\frac12,\bar 0}\phi),\\
r_{i-\frac12,\bar 1}&=r_{i-\frac12}+\sum_{k\geq 1}u^k(r_{i+k-\frac12}+r_{i+k}\frac{\partial}{\partial\phi}),\\
&=r_{i-\frac12}+u(r_{i+1-\frac12,\bar 1}+r_{i+1,\bar 1}\frac{\partial}{\partial\phi}).
\end{aligned}\tag{3.12}$$

Let X be an even matrix. Because of the multiplicative property of Berezinians we can add multiples of a row to another row of X without changing $\operatorname{ber}(X)$ and $\operatorname{ber}^*(X)$. Using such row operations we see, using (3.12), that

$$\operatorname{ber}\left([Q_{\bar 0}(u,\phi)\mathcal{W}]_-\right)=\operatorname{ber}\left(A_0(r_{0,\bar 0})\right),$$
$$\operatorname{ber}^*\left([Q_{\bar 1}(u,\phi)\mathcal{W}]_-\right)=\operatorname{ber}^*\left(A_{-\frac12}(r_{-\frac12,\bar 1})\right).$$

Now ber is linear in even rows, and ber* in odd rows, so by (3.12) we find

$$\operatorname{ber}([Q_{\bar 0}(u,\phi)\mathcal{W}]_-) = \operatorname{ber}(A) + \sum_{i>0} u^i[\operatorname{ber}(A_0(r_i)) + \operatorname{ber}(A_0(r_{i-\frac12}))\phi],$$

and

$$\operatorname{ber}^*([Q_{\bar 1}(u,\phi)\mathcal{W}]_-) = \operatorname{ber}^*(A) + + \sum_{i>0} u^i[\operatorname{ber}^*(A_{-\frac12}(r_{i-\frac12})) + \operatorname{ber}^*(A_{-\frac12}(r_i))\frac{\partial}{\partial\phi}].$$

Comparing with (3.11) proves the lemma. □

We now consider the flow on $\mathcal{S}$gr generated by the negative part of the Jacobian Heisenberg algebra: define

$$\gamma(t) = \exp(\sum_{i>0} t_i z^{-i} + t_{i-\frac12} z^{-i}\theta), \quad t_i \in \Lambda_{\mathrm{ev}}, t_{i-\frac12} \in \Lambda_{\mathrm{odd}} \tag{3.13}$$

and put for $W \in \mathcal{S}$gr:

$$W(t) = \gamma(t)^{-1}W.$$

The τ-functions associated to a point W in the big cell are then functions on $\mathcal{J}\mathrm{Heis}_{-,\mathrm{ev}}$:

$$\tau_W(t), \tau^*_W(t) : \mathcal{J}\mathrm{Heis}_{-,\mathrm{ev}} \to \Lambda \cup \{\infty\} \tag{3.14}$$

given by

$$\tau_W(t) = \frac{\sigma(\gamma(t)^{-1}W)}{\gamma(t)^{-1}\sigma(W)} = \frac{\operatorname{ber}([\gamma(t)^{-1}\circ\mathcal{W}]_-)}{\operatorname{ber}([\mathcal{W}]_-)}, \tag{3.15}$$

$$\tau^*_W(t) = \frac{\sigma^*(\gamma(t)^{-1}W)}{\gamma(t)^{-1}\sigma^*(W)} = \frac{\operatorname{ber}^*([\gamma(t)^{-1}\circ\mathcal{W}]_-)}{\operatorname{ber}^*([\mathcal{W}]_-)}. \tag{3.16}$$

Here σ and σ^* are the sections of $\mathrm{Ber}^*(\mathcal{S}\mathrm{gr})$ and $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ defined in (3.3) and $\gamma^{-1} \in Gl_{\infty|\infty}(\Lambda)$ acts via (3.6) on $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ and $\mathrm{Ber}^*(\mathcal{S}\mathrm{gr})$.

The Baker function of W becomes now a function on $\mathcal{J}\mathrm{Heis}_{-,\mathrm{ev}}$, and we have an expression in terms of a quotient of (shifted) τ-functions:

$$w_{\bar 0}(t;u,\phi) = \frac{\tau_W(t;Q_{\bar 0})}{\tau_W(t)}, \quad w_{\bar 1}(t;u,\phi) = \frac{\tau^*_W(t;Q_{\bar 1})}{\tau^*_W(t)}, \tag{3.17}$$

where

$$\tau_W(t;Q_{\bar 0}) = \frac{\operatorname{ber}([Q_{\bar 0}\mathcal{W}]_-)}{\operatorname{ber}(\mathcal{W}_-)}, \quad \tau^*_W(t;Q_{\bar 1}) = \frac{\operatorname{ber}^*([Q_{\bar 1}\mathcal{W}]_-)\phi}{\operatorname{ber}^*(\mathcal{W}_-)}.$$

Note that even if we are only interested in the Jacobian Heisenberg flows the full Heisenberg flows automatically appear in the theory if we express the Baker functions in terms of the τ functions. In principle we could also consider the flows on $\mathcal{S}$gr generated by the full super Heisenberg algebra $\mathcal{S}$Heis. However, since $\mathcal{S}$Heis is non–Abelian the interpretation of these flows is less clear and therefore we leave the discussion of these matters to another occasion.

4. The Krichever map and algebro-geometric solutions

4.1. **The Krichever map.** Consider now a set of geometric data $(X, P, (z, \theta), \mathcal{L}, t)$, where:

- X is a generic SKP curve as before.
- P is an irreducible divisor on X, so that $P^{\rm red}$ is a single point of the underlying Riemann surface $X^{\rm red}$.
- (z, θ) are local coordinates on X near P, so that P is defined by the equation $z = 0$.
- $\mathcal{L}$ is an invertible sheaf on X.
- t is a trivialization of $\mathcal{L}$ in a neighborhood of P, say $U_P = \{|z^{\rm red}| < 1\}$.

We will associate to this data a point of the super Grassmannian $\mathcal{S}$gr.

For studying meromorphic sections of $\mathcal{L}$ we have the exact sequence

$$0 \to \mathcal{L} \stackrel{\rm inc}{\to} \mathcal{L}(P) \stackrel{\rm res}{\to} \mathcal{L}_{P^{\rm red}} \cong \Lambda|\Lambda \to 0, \tag{4.1}$$

which gives

$$H^0(\mathcal{L}) \hookrightarrow H^0(\mathcal{L}(P)) \to \Lambda|\Lambda \to H^1(\mathcal{L}) \to H^1(\mathcal{L}(P)) \to 0, \tag{4.2}$$

where the residue is the pair of coefficients of z^{-1} and θz^{-1} in the Laurent expansion.

Let $\mathcal{L}(*P) = \lim_{n\to\infty} \mathcal{L}(nP)$ be the sheaf of sections of $\mathcal{L}$ holomorphic except possibly for a pole of arbitrary order at P. The *Krichever map* associates to a set of geometric data as above the Λ-module of formal Laurent series $W = z\, t[H^0(X, \mathcal{L}(*P))]$, which will be viewed as a submodule of H.

In [MR91, Rab91] the concern was expressed that W might not be freely generated, and hence not an element of $\mathcal{S}$gr as we have defined it. However,

Theorem 4.1.1. *$H^0(X, \mathcal{L}(*P))$ is a freely generated Λ-module, and $W \in \mathcal{S}gr$. Further, W belongs to the big cell if the geometric data satisfy $H^0(X, \mathcal{L}) = H^1(X, \mathcal{L}) = 0$, which happens generically if* $\deg \mathcal{L} = g - 1$.

Proof. Assume first that $H^0(X,\mathcal{L}) = H^1(X,\mathcal{L}) = 0$. Then the sequence (4.2) applied to $\mathcal{L}$ gives

$$0 \to H^0(\mathcal{L}(P)) \to \Lambda|\Lambda \to 0 \to H^1(\mathcal{L}(P)) \to 0, \tag{4.3}$$

so that $H^0(X,\mathcal{L}(P))$ is freely generated by an even and odd section having principal parts z^{-1} and θz^{-1}, and $H^1(X,\mathcal{L}(P))$ is still zero. Applying the same sequence inductively to $\mathcal{L}(nP)$ shows that $H^0(X,\mathcal{L}(*P))$ is freely generated by one even and one odd section of each positive pole order. So W is obtainable from H_- by multiplication by a lower triangular invertible matrix, and W belongs to the big cell of $\mathcal{S}$gr. We also have $H^i(\mathcal{L}^{\,\mathrm{split}}) = 0, i = 0,1$, from Theorem 2.6.1. And, by the super Riemann-Roch Theorem (2.22), $\deg \mathcal{L} = \deg \mathcal{L}^{\,\mathrm{split}} = g-1$. Moreover, by semicontinuity, in $\mathrm{Pic}^{g-1}(X)$ the cohomology groups $H^i(\mathcal{L})$ can only get larger on Zariski closed subsets, so generically they are zero.

Now consider the general situation in which $H^i(\mathcal{L})$ may not be zero. Still, by twisting, $H^1(\mathcal{L}(nP)) = H^1(\mathcal{L}^{\,\mathrm{split}}(nP)) = 0$ for n sufficiently large. Then, by the previous argument, $H^0(\mathcal{L}(*P))$ has non purely nilpotent elements with poles of order $n+1$ and higher; the worry is that one may only be able to find nilpotent generators for $H^0(\mathcal{L}(nP))$. So take $f \in H^0(\mathcal{L}(nP))$ of order k in nilpotents: its image in $H^0(\mathcal{L}(nP)/\mathfrak{m}^k)$ is zero, but its image $\hat{f}$ in $H^0(\mathcal{L}(nP)/\mathfrak{m}^{k+1})$ is nonzero and also lies in $\Lambda^k H^0(\mathcal{L}^{\,\mathrm{split}}(nP))$. Then $\hat{f}$ can be identified with a sum of elements f_a of $H^0(\mathcal{L}^{\,\mathrm{split}}(nP))$ with coefficients from Λ^k. By the extension sequence (2.21), each f_a can be extended order by order in nilpotents to an element of $H^0(\mathcal{L}(nP))$ which is not purely nilpotent. So we can write the order k element f as a Λ-linear combination of not purely nilpotent elements of $H^0(\mathcal{L}(nP))$, modulo an element of order $k+1$. Induction on k shows then that any element of $H^0(\mathcal{L}(nP))$ is a Λ-linear combination of not purely nilpotent elements of $H^0(\mathcal{L}(nP))$. So there exists a set of non-nilpotent elements which span $H^0(\mathcal{L}(nP))$ over Λ. A linearly independent subset of these completes a basis for $H^0(\mathcal{L}(*P))$. □

4.2. **The Chern class of the Ber bundle on** $\mathrm{Pic}^0(X)$**.** By the arguments of the previous subsection we have, in case $W \in \mathcal{S}\mathrm{gr}$ is obtained by the Krichever map from geometric data $(X,P,\mathcal{L},(z,\theta),t)$, an exact sequence of Λ-modules:

$$0 \to H^0(X,\mathcal{L}) \to W \to H_- \to H^1(X,\mathcal{L}) \to 0. \tag{4.4}$$

We can interpret the Ber bundle $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ in terms of this sequence as follows. Let M be a free Λ-module, possibly of infinite rank, and let $B = \{\mu\}$ be a collection of bases for M such that any two bases

$\mu, \mu' \in B$ are related by $\mu' = \mu T$ where $T \in \mathrm{Aut}(M)$ has a well defined Berezinian. Then we associate to the pair (M, B) a free rank $(1 \mid 0)$ module $\mathrm{ber}(M)$ with generator $b(\mu)$ for any $\mu \in B$ with identification $b(\mu') = \mathrm{ber}(T)b(\mu)$.

The fiber of $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ at W can then be interpreted as $\mathrm{ber}(W)$, using the collection of admissible bases as B in the above definition. Similarly we can construct on $\mathcal{S}\mathrm{gr}$ a line bundle with fiber at W the module $\mathrm{ber}(H_-)$. Clearly this bundle is trivial, so we can, even better, think of $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ as having fiber $\mathrm{ber}(W) \otimes \mathrm{ber}^*(H_-)$. But by the properties of the Berezinian we get from (4.4)

$$\mathrm{ber}(W) \otimes \mathrm{ber}^*(H_-) = \mathrm{ber}(H^0(X, \mathcal{L})) \otimes \mathrm{ber}^*(H^1(X, \mathcal{L})).$$

Now we have seen in subsection 2.14 that the Ber bundle $\mathrm{Ber}(\mathrm{Pic}^0(X))$ on $\mathrm{Pic}^0(X)$ has the same fiber, with the difference that there we were dealing with bundles of degree 0 and here $\mathcal{L}$ has degree $g - 1$.

For fixed $(X, P, (z, \theta))$ the collection M consisting of Krichever data $(X, P, (z, \theta), \mathcal{L}, t)$ forms a supermanifold and we have two morphisms

$$i : M \to \mathcal{S}\mathrm{gr}, \quad p : M \to \mathrm{Pic}^0(X)$$

where i is the Krichever map and p is the projection from Krichever data to the line bundle $\mathcal{L}$. (Here we identify $\mathrm{Pic}^n(X)$ with $\mathrm{Pic}^0(X)$ via the invertible sheaf $\mathcal{O}_X(-nP)$.) Then we see that $i^*(\mathrm{Ber}(\mathcal{S}\mathrm{gr})) \simeq p^*(\mathrm{Ber}(\mathrm{Pic}^0(X)))$. This fact allows us to prove Theorem 2.14.1.

Note first that we have a surjective map

$$\mathcal{J}\mathrm{Heis}_- \to H^1(X, \mathcal{O}_X). \tag{4.5}$$

Indeed, let $X = U_0 \cup U_P$ be an open cover where $U_0 = X - P^{\mathrm{red}}$ and U_P is a suitable disk around P^{red}. Then if $[a] \in H^1(X, \mathcal{O}_X)$ is represented by $a \in \mathcal{O}_X(U_0 \cap U_P)$ we can write, using the local coordinates on U_P, $a = a_P + \sum_{i>0} a_i z^{-i} + \alpha_i z^{-i}\theta$, with $a_P \in \mathcal{O}_X(U_P)$. Then $a - a_P = \sum a_i z^{-i} + \alpha_i z^{-i}\theta \in \mathcal{J}\mathrm{Heis}_-$ and $[a] = [a - a_P]$. Now the tangent space to any point $\mathcal{L} \in \mathrm{Pic}^0(X)$ can be identified with $H^1(X, \mathcal{O}_X)$ and so we have a surjective map from $\mathcal{J}\mathrm{Heis}_-$ to the tangent space of $\mathrm{Pic}^0(X)$. Note secondly that a change of trivialization of $\mathcal{L}$, given by $t \mapsto t'$, corresponds to multiplication of the point $W \in \mathcal{S}\mathrm{gr}$ by an element $a_0 + \alpha_0\theta + \sum_{i>0} a_i z^i + \alpha_i z^i \theta$ of the group corresponding to $\mathcal{J}\mathrm{Heis}_+$. From these two facts we conclude that there is a surjective map from $\mathcal{J}\mathrm{Heis}$ to the tangent space to the image of the Krichever map $i : M \to \mathcal{S}\mathrm{gr}$ at any point $W = W(X, P, (z, \theta), \mathcal{L}, t)$. Now the first Chern class of $\mathrm{Ber}(\mathcal{S}\mathrm{gr})$ is calculated from the cocycle (3.5) on $gl_{\infty|\infty}(\Lambda)$ and it follows from Lemma 3.3.1 that the restriction of this cocycle to $\mathcal{J}$Heis is identically zero. This implies that

$$i^*(c_1[\mathrm{Ber}(\mathcal{S}\mathrm{gr})]) = p^*(c_1[\mathrm{Ber}(\mathrm{Pic}^0(X))]) = 0.$$

But the map $p : M \to \mathrm{Pic}^0(X)$ is surjective, so we finally find that $c_1(\mathrm{Ber}(\mathrm{Pic}^0(X))) = 0$ and $\mathrm{Ber}(\mathrm{Pic}^0(X))$ is topologically trivial, proving Theorem 2.14.1.

4.3. **Algebro-geometric tau and Baker functions.** We consider geometric data mapping to W in the big cell of $\mathcal{S}$gr, so that $\deg \mathcal{L} = g-1$. As discussed in Section 3, we can associate to W both a tau function and a Baker function. A system of super KP flows on $\mathcal{S}$gr applied to W produces an orbit corresponding to a family of deformations of the original geometric data. The simplest system of super KP flows, the "Jacobian" system of Mulase and Rabin [Mul90, Rab91], deforms the geometric data by moving $\mathcal{L}$ in $\mathrm{Pic}^{g-1}(X)$. Solutions to this system for X a super elliptic curve were obtained in terms of super theta functions in [Rab95b]. On the basis of the ordinary KP theory, cf. [SW85], section 9, we might expect that in general the tau and Baker functions for this family can be given explicitly as functions of the flow parameters by means of the super theta functions (when these exist) on the Jacobian of X. We now discuss the extent to which this is possible.

Recall from (4.5) that we have a surjection from $\mathcal{J}\mathrm{Heis}_{-,\mathrm{ev}}$ to the cohomology group $H^1(X, \mathcal{O}_{X,\mathrm{ev}})$. By exponentiation we obtain a map from $\mathcal{J}\mathrm{Heis}_{-,\mathrm{ev}}$ to $\mathrm{Pic}^0(X)$ and these maps fit together in a diagram

(4.6)

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & H^1(X,\mathbb{Z}) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & K_0 & \longrightarrow & \mathcal{J}\mathrm{Heis}_{-,\mathrm{ev}} & \longrightarrow & H^1(X,\mathcal{O}_{X,\mathrm{ev}}) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & \mathcal{J}\mathrm{Heis}_{-,\mathrm{ev}} & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & K/K_0 & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Here K_0 is the $\Lambda_{\rm ev}$-submodule of elements f of $\mathcal{J}\mathrm{Heis}_{-,\rm ev}$ that split as $f = f_0 + f_P$, with $f_0 \in \mathcal{O}_X(U_0)$ and $f_P \in \mathcal{O}_X(U_P)$ and K is the Abelian subgroup (not submodule!) of elements k of $\mathcal{J}\mathrm{Heis}_{-,\rm ev}$ that after exponentiation factorize: $e^k = \phi_k e^{k_p}$, with $\phi_k \in \mathcal{O}_X(U_0)^\times$ and $k_P \in \mathcal{O}_X(U_P)$. From the Snake Lemma it then follows that $H^1(X,\mathbb{Z}) \simeq K/K_0$. So a function $\hat{F}$ on $\mathcal{J}\mathrm{Heis}_{-,\rm ev}$ descends to a function F on $H^1(X, \mathcal{O}_{X,\rm ev})$ if it is invariant under K_0. The automorphic behavior of such a function F with respect to the lattice $H^1(X,\mathbb{Z})$ translates into behaviour of $\hat{F}$ under shifts by elements of K. In particular we consider the function τ_W associated to a point W in the big cell of $\mathcal{S}\mathrm{gr}$, see (3.14) and (3.15). This is a function on $\mathcal{J}\mathrm{Heis}_{-,\rm ev}$ and, because of Lemma 3.3.1, we see by an easy adaptation of the proof of Lemma 9.5 in [SW85] that

$$\tau_W(f+k) = \tau_W(f)\tau_W(k), \quad f \in \mathcal{J}\mathrm{Heis}_{-,\rm ev}, k \in K.$$

In particular we obtain by restriction a homomorphism

$$\tau_W : K_0 \to \Lambda_{\rm ev}^\times.$$

Let $\eta : K_0 \to \Lambda_{\rm ev}$ be a homomorphism such that $\tau_W(k_0) = e^{\eta(k_0)}$, for all $k_0 \in K_0$. Then we can define a new function

$$\hat{\tau}_1(f) = \tau_W(f)e^{-\eta(f)}.$$

Then $\hat{\tau}_1(k_0) = 1$, but still we have

$$\hat{\tau}_1(f+k) = \hat{\tau}_1(f)\hat{\tau}_1(k), \tag{4.7}$$

so that $\hat{\tau}_1$ descends to a function τ_1 on $H^1(X, \mathcal{O}_{X,\rm ev})$. From (4.7) we see that τ_1 corresponds to a (meromorphic) section of a line bundle on $\mathrm{Pic}^0(X)$ with trivial Chern class.

A suitable ratio of translated theta functions gives a section of this same bundle, so that τ_1 is expressed as this ratio times a meromorphic function, the latter being rationally expressible in terms of super theta functions. Then the modified tau function τ_1 is rationally expressed in terms of super theta functions.

The even Baker function $w_{\bar{0}}^W(z,\theta)$ associated to the point W is just the even section of $\mathcal{L}$ holomorphic except for a pole $1/z$ at P. Such a section can be specified by its restrictions to the charts U_0 and U_P. The Jacobian super KP flows act by multiplying the transition function of $\mathcal{L}$ across the boundary of U_P by a factor $\gamma(t)$ as in (3.13). The corresponding action on the associated point W of $\mathcal{S}\mathrm{gr}$ is generated by the matrices $\lambda(n)+\mu(n)$ and $f(n)$ of Section 3; the remaining matrices generate deformations of the curve X and enter the Kac–van de Leur SKP flows. Then $w_{\bar{0}}^{W(t)}/w_{\bar{0}}^W$ is a section of the bundle with transition function (3.13). Equivalently, it is a meromorphic function on U_0 which extends into U_P except for an essential singularity of the form (3.13), having zeros at the divisor of $\mathcal{L}(t)$ and poles at the divisor of $\mathcal{L}$. By analogy with the "Russian formula" of ordinary KP theory, such a function would be expressed in the form

$$\exp[\sum_{k=1}^{\infty}\int_{(0,0)}^{(z,\theta)}(t_k\hat{\psi}_k + t_{k-\frac{1}{2}}\hat{E}_k) + c(t)] \tag{4.8}$$

times a ratio of theta functions providing the zeros and poles. Here $\hat{\psi}_k$ and $\hat{E}_k$ are differentials on $\hat{X}$, with vanishing a-periods and holomorphic except for the behavior near P,

$$\hat{\psi}_k \sim \hat{D}(z^{-k}) = -k\rho\hat{z}^{-(k+1)}, \quad \hat{E}_k \sim \hat{D}(\theta z^{-k}) = \hat{z}^{-k}. \tag{4.9}$$

The constant $c(t)$ is linear in the flow parameters. In addition to the symmetry of the period matrix, we have to require the existence of these differentials. This requires that they exist in the split case, and then that these split differentials extend through the sequence (2.21). In the split case, the odd differentials $\hat{\psi}_k$ are just θ times the ordinary differentials on the reduced curve which appear in the Russian formula (and they do extend). However, the even differentials $\hat{E}_k$ are sections of $\mathcal{N}$, which is of degree zero and nontrivial, with $h^1 = g-1$. Consequently, when $g > 1$ there will be Weierstrass gaps in the list of pole orders of these differentials. This means that the odd flow parameters corresponding to the missing differentials must be set to zero in order for the Baker function to assume the "Russian" form. Even then, however, the function given by the Russian formula will generically behave as $1 + \alpha\theta + \mathcal{O}(z)$ for $z \to 0$, rather than the correct $1 + O(z)$ for $w_{\bar{0}}^{W(t)}$ containing no θ/z pole. In [Rab95b] this was dealt with by including a term $\xi\hat{E}_0$ in the exponential, taking ∂_ξ to construct a section with a pure θ/z pole, and subtracting off the appropriate multiple of this. In general, however, no such $\hat{E}_0$ will exist. These difficulties are understandable in view of the relations (3.17) which require that the tau function be known for the full set of K-vdL flows in order to compute the Baker functions for even the Jacobian flows. Since the dependence of the tau function on the non-Jacobian flows is likely to be far more complicated than our super theta functions, it is unlikely that the Baker functions can be expressed in terms of them.

Appendix A. Duality and Serre duality

Let Λ be our usual ground ring $\mathbb{C}[\beta_1, \ldots, \beta_n]$ with the β_i odd indeterminates. In this Appendix we will discuss duality for Λ-modules (cf. chapter 21 in Eisenbud, [Eis95] for the case of commutative rings). Then we will use this to extend Serre duality for supermanifolds over the ground field $\mathbb{C}$ ([HW87], [OP84]) to supermanifolds over the ground ring Λ.

Finally we discuss the more explicit form of Serre duality that one has in case of super curves. One way of proving Serre duality for super curves over general Λ would be by using the properties of the supertrace in infinite rank Λ-modules to define a residue, generalizing the method of Tate, [Tat68]. However, we have available in our case contour integration which also provides us with a residue map.

A.1. **Duality of Λ-modules.** Let M be an object of $\mathcal{M}\mathrm{od}_\Lambda$, the category of finitely generated Λ-modules. We give the $\mathbb{Z}_2$-graded vector space $E(M) = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ the structure of a Λ-module via

$$\lambda \cdot \phi(m) = (-1)^{|\lambda||\phi|}\phi(\lambda m),$$

for all $m \in M$ and all homogeneous $\lambda \in \Lambda$ and $\phi \in E(M)$. Then one checks that $E = E(-)$ is a *dualizing functor* on $\mathcal{M}\mathrm{od}_\Lambda$: it is contravariant, exact, Λ-linear and satisfies $E^2 \simeq 1_{\mathcal{M}\mathrm{od}_\Lambda}$. One also checks that $E(\Lambda) \simeq \Lambda$, up to a possible parity change. In the sequel we will ignore these parity changes.

An explicit (basis dependent) isomorphism $\Lambda \to \mathrm{Hom}_{\mathbb{C}}(\Lambda, \mathbb{C})$ can be described as follows. The monomials $\beta^I = \beta_1^{i_1} \dots \beta_n^{i_n}$, $i_j = 0, 1$, form a basis of Λ as $\mathbb{C}$-vector space, and we let ϕ_{β^I} be the dual basis of $E(\Lambda) = \mathrm{Hom}_{\mathbb{C}}(\Lambda, \mathbb{C})$, so that $\phi_{\beta^I}(\beta^J) = \delta_{IJ}$. Then the Λ-homomorphism that maps $1 \in \Lambda$ to the linear functional $\phi_{\beta_1 \dots \beta_n}$ is an isomorphism, odd in case n is odd.

From the fact that E is a dualizing functor we see that the map $\mathrm{Hom}_\Lambda(M, N) \to \mathrm{Hom}_\Lambda(E(N), E(M))$ is an isomorphism for all objects M, N and hence

$$E(M) = \mathrm{Hom}_\Lambda(\Lambda, E(M)) \simeq \mathrm{Hom}_\Lambda(M, E(\Lambda)) = \mathrm{Hom}_\Lambda(M, \Lambda).$$

In other words, up to a possible parity switch, we can identify (functorially) the Λ-modules of $\mathbb{C}$-linear and of Λ-linear homomorphisms on any finitely generated Λ-module M:

$$\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}) \simeq \mathrm{Hom}_\Lambda(M, \Lambda)$$

We will identify the two and use for both the symbol M^*.

Note that the above implies that the functor that maps M to $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$ on $\mathcal{M}\mathrm{od}_\Lambda$ (naturally isomorphic to the functor E) is exact: exact sequences get mapped to exact sequences. Equivalently: Λ is injective as a module over itself. Also note that the double dual of M is isomorphic to M itself.

A.2. **Serre duality of supermanifolds.** Serre duality for supermanifolds $(Y, \mathcal{O}_Y)$ of dimension $(p \mid q)$ over the complex numbers is discussed in Haske and Wells [HW87] and in Ogievetsky and Penkov [OP84].

A *dualizing sheaf* on $(Y, \mathcal{O}_Y)$ is an invertible sheaf ω_Y together with a fixed homomorphism

$$t : H^p(X, \omega_Y) \to \mathbb{C}$$

such that the induced pairing for all $\mathcal{F}$

$$H^i(Y, \mathcal{F}^* \otimes \omega_Y) \otimes H^{p-i}(Y, \mathcal{F}) \xrightarrow{t} \mathbb{C}$$

gives an isomorphism, called Serre duality,

$$H^i(Y, \mathcal{F}^* \otimes \omega_Y) \xrightarrow{\sim} H^{p-i}(Y, \mathcal{F})^*. \tag{A.1}$$

Note that this is an isomorphism of $\mathbb{Z}_2$-graded vector spaces over $\mathbb{C}$. Dualizing sheaves are unique, up to isomorphism.

If $\mathcal{T}_Y$ is the tangent sheaf of $(Y, \mathcal{O}_Y)$ with transition functions $J_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(p \mid q, \mathbb{C})$, then the *Berezinian sheaf* $\mathcal{B}\mathrm{er}_Y$ is the invertible sheaf with transition functions $\mathrm{ber}\,(J_{\alpha\beta}) : U_\alpha \cap U_\beta \to \mathbb{C}^\times$. In [HW87], [OP84] it is proved that $\mathcal{B}\mathrm{er}_Y$ is a dualizing sheaf.

We are interested in the relative situation: supermanifolds $(Z, \mathcal{O}_Z) \to (\bullet, \Lambda)$ of dimension $(p \mid q)$ over Λ. The structure sheaf $\mathcal{O}_Z$ contains n independent global odd constants. We can reduce this to the absolute case by thinking of $(Z, \mathcal{O}_Z)$ as a supermanifold $(Z, \mathcal{O}_Y)$ of dimension $(p \mid q + n)$ over $\mathbb{C}$: the constants β_i generating Λ are now interpreted as coordinates.

The tangent sheaf $\mathcal{T}_Y$ of derivations of $\mathcal{O}_Y$ has then global sections $\frac{\partial}{\partial \beta_I}$, in contrast to the relative tangent sheaf $\mathcal{T}_{Z/\Lambda}$. One sees easily, however, that the invertible sheaf $\mathcal{B}\mathrm{er}_Y$, constructed from the tangent sheaf $\mathcal{T}_Y$, is the same as $\mathcal{B}\mathrm{er}_{Z/\Lambda}$, constructed from the relative tangent sheaf $\mathcal{T}_{Z/\Lambda}$.

Now let $\mathcal{F}$ be a coherent, locally free sheaf of $\mathcal{O}_Z$-modules. Then the associated cohomology groups are finitely generated Λ-modules, so we can use the theory of Appendix A.1. In particular we see that there is a homomorphism

$$t : H^p(Z, \mathcal{B}\mathrm{er}_{Z/\Lambda}) \to \Lambda \tag{A.2}$$

inducing an isomorphism of Λ-modules, also called Serre duality,

$$H^i(Z, \mathcal{F}^* \otimes \mathcal{B}\mathrm{er}_{Z/\Lambda}) \xrightarrow{\sim} H^{p-i}(Z, \mathcal{F})^*, \tag{A.3}$$

where now $H^p(Z, \mathcal{F})^*$ means the Λ-linear dual.

In case of $N = 1, 2$ super curves $(X, \mathcal{O}_X) \to (\bullet, \Lambda)$ we can be more explicit about the homomorphism t in (A.2). Write $\mathcal{B}\mathrm{er}_X$ for the relative Berezinian $\mathcal{B}\mathrm{er}_{X/\Lambda}$. The cohomology of $\mathcal{B}\mathrm{er}_X$ is calculated by the sequence

$$0 \to H^0(X, \mathcal{B}\mathrm{er}_X) \to \mathrm{Rat}(\mathcal{B}\mathrm{er}_X) \to \mathrm{Prin}(\mathcal{B}\mathrm{er}_X) \to H^1(X, \mathcal{B}\mathrm{er}_X) \to 0,$$

where $\mathrm{Rat}(\mathcal{B}\mathrm{er}_X)$ are the rational sections and $\mathrm{Prin}(\mathcal{B}\mathrm{er}_X)$ the principal parts. On $\mathrm{Prin}(\mathcal{B}\mathrm{er}_X)$ we have for every $x \in X$ a residue map $\mathrm{Res}_x\omega \mapsto \frac{1}{2\pi i}\oint_{C_x}\omega$, where C_x is a contour around x, using the integration on $N = 1, 2$ curves introduced in subsection 2.4. Then $t = \sum_{x\in X}\mathrm{Res}_x$ and the classical residue theorem holds: if $\omega \in \mathrm{Rat}(\mathcal{B}\mathrm{er}_X)$ then $t(\omega) = 0$. Now that we have the residue we can proceed to prove Serre duality as in the classical case, cf. [Ser88].

Appendix B. Real structures and conjugation.

Let $\Lambda = \mathbb{C}[\beta_1 \dots, \beta_n]$ be the Grassmann algebra generated by n odd indeterminates. Choose a sign $\epsilon = \pm 1$ and define a real structure on Λ for this choice as a real-linear, even map $\omega : \Lambda \to \Lambda$ such that

1. $\omega(ca) = \bar{c}\omega(a), \quad c \in \mathbb{C}, a \in \Lambda,$
2. $\omega^2 = 1.$
3. $\omega(\beta_i) = \beta_i,$
4. $\omega(ab) = \epsilon^{|a|\,|b|}\omega(b)\omega(a),$

where a, b are homogeneous elements of parity $|a|, |b|$. We will often write $\bar{a}$ for $\omega(a)$.

A real structure induces a decomposition of Λ into eigenspaces, $\Lambda = \Lambda_{\Re} \oplus \Lambda_{\Im}$, where $\Lambda_{\Re}$ is the $+1$ eigenspace and $\Lambda_{\Im}$ the -1 eigenspace for ω. Multiplication by i is an isomorphism, so we get $\Lambda = \Lambda_{\Re} \oplus i\Lambda_{\Re}$.

Appendix C. Linear equations in the super category.

C.1. **Introduction.** Cramer's rule tells us that a system of linear equations over a commutative ring R:

$$xA = y, \tag{C.1}$$

with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ n-component row vectors and A an $n \times n$ matrix with coefficients in R, can be solved by quotients of determinants:

$$x_i = \frac{\det(A_i(y))}{\det(A)}, \quad i = 1, \dots, n, \tag{C.2}$$

where $A_i(y)$ is the matrix obtained by replacing in A the ith row by the row vector y.

We want to study (C.1) in the super category. So we fix a decomposition $n = k + l$. Call an index $i \in \{1, \dots, n\}$ *even* if $i \le k$ and *odd* otherwise. Consider an even matrix $A = (a_{ij})_{i,j=1}^n$ over Λ: a_{ij} is an element of the even part Λ_{ev} if i and j have the same parity and of the odd part $\Lambda_{\rm odd}$ otherwise. Note that it is not necessary to specify the parity of y in (C.1): it can be even, odd or inhomogeneous.

In the theory of linear algebra over a Grassmann algebra Λ in many (but not all) respects a role analogous to that of the determinant in the commutative case is played by the *Berezinian* and its inverse: for an even matrix A as above we define

$$\begin{aligned} \text{ber}\,(A) &= \det\,(X - \alpha Y^{-1}\beta)\det\,(Y^{-1}), \\ \text{ber}^*(A) &= \det\,(X^{-1})\det\,(Y - \beta X^{-1}\alpha) = \frac{1}{\text{ber}(A)} \end{aligned} \tag{C.3}$$

We will discuss how Cramer's rule can be generalized in this setting. The result is as follows: the solution of (C.1) for A an even super matrix is given by

$$x_i = \begin{cases} \text{ber}\,(A_i(y))/\text{ber}\,(A) & \text{if } i \le k, \\ \text{ber}^*\,(A_i(y))/\text{ber}^*\,(A) & \text{if } i > k. \end{cases} \tag{C.4}$$

Note that in general the matrix $A_i(y)$ is not even. However, in ber (A) for $i \le k$ (resp. ber* (A) for $i > k$) the entries a_{ij}, $j = 1, \dots, n$ of row i occur only linearly. So, if ber (A) (resp. ber* (A)) exists, we mean by ber $(A_i(y))$ for $i \le k$ (resp. ber* $(A_i(y))$ for $i > k$) the element of Λ obtained by replacing in ber (A) (resp. ber* (A)) all a_{ij} by y_j for $j = 1, \dots, n$.

The expression for the entries in even positions, i.e., x_i, $i \le k$, can be found in [UYI89], but we haven't seen the solution for the odd positions in the literature. The main ingredient is the Gelfand-Retakh theory of quasi determinants [GR91, GR93].

C.2. **Quasideterminants.** Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with entries independent (non commuting) variables a_{ij}. Let A^{ij} be the submatrix obtained by deleting in A row i and column j. Following Gelfand-Retakh we introduce n^2 *quasideterminants* $|A|_{ij}$, rational expressions in the variables a_{pq}. If $n = 1$ we put $|A|_{11} = a_{11}$ and for $n > 1$ we define recursively

$$|A|_{ij} = a_{ij} - \sum_{\substack{p \ne j \\ q \ne i}} a_{ip} |A^{ij}|_{qp}^{-1} a_{qj}. \tag{C.5}$$

If we assume that the variables a_{ij} commute amongst themselves, then we have

$$|A|_{ij} = (-1)^{i+j} \det\,(A)/\det\,(A^{ij}). \tag{C.6}$$

Returning to the general case, one proves that the inverse of A, as a matrix over the ring of rational functions in the a_{ij}, is the matrix

$B = (b_{ij})$, where $b_{ij} = |A|_{ji}^{-1}$. This means that we can also define the quasideterminant as

$$|A|_{ij} = a_{ij} - \sum_{\substack{p \neq j \\ q \neq i}} a_{ip} c_{pq}^{(ij)} a_{qj}, \tag{C.7}$$

where $C^{(ij)} = (c_{pq}^{(ij)})$ is the inverse to the matrix A^{ij}. This second definition is useful if one wants to think of the entries of A as elements of a ring R. In that case it is perfectly well possible that the inverse $C^{(ij)}$ of A^{ij} exists, but that some entry $c_{pq}^{(ij)}$ is not invertible in R. In this situation (C.7) allows us to define the quasideterminant $|A|_{ij}$, whereas (C.5) might make no sense.

Quasideterminants have the following properties:

P1: If the matrix B is obtained from A by multiplying row i from the left by λ (a new independent variable) then for all $j = 1, 2, \dots, n$ we have

$$|B|_{ij} = \lambda |A|_{ij}, \quad |B|_{kj} = |A|_{kj}, k \neq i.$$

Similarly if the matrix C is obtained from A by multiplying column j from the right by μ we have for all $i = 1, \dots, n$:

$$|C|_{ij} = |A|_{ij}\mu, \quad |C|_{ik} = |A|_{ik}, k \neq j.$$

P2: If B is obtained from A by adding row k to some other row, then for all $j = 1, \dots, n$:

$$|B|_{ij} = |A|_{ij}, \quad k \neq i.$$

If C is obtained from A by adding column k to some other column, then for all $i = 1, \dots, n$:

$$|C|_{ij} = |A|_{ij}, \quad k \neq j.$$

C.3. **Restriction to super algebra.** We keep the decomposition $n = k + l$ and we consider an even matrix $A = (a_{ij})_{i,j=1}^n$ over supercommuting variables: a_{ij} is an even variable if i and j have the same parity and an odd variable otherwise.

Lemma C.3.1. *Let A be an even matrix as above. Then*

- *If i, j are both even then*

$$|A|_{ij} = (-1)^{i+j} \operatorname{ber}(A) / \operatorname{ber}(A^{ij}).$$

- *If i, j are both odd then*

$$|A|_{ij} = (-1)^{i+j} \operatorname{ber}^*(A) / \operatorname{ber}^*(A^{ij}).$$

Proof. We can write

$$A = \begin{pmatrix} X & \alpha \\ \beta & Y \end{pmatrix} = \begin{pmatrix} 1 & \alpha Y^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X - \alpha Y^{-1}\beta & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y^{-1}\beta & 1 \end{pmatrix}.$$

This shows that A is obtained from $\begin{pmatrix} X - \alpha Y^{-1}\beta & 0 \\ 0 & Y \end{pmatrix}$ by operations that don't change the quasideterminant $|A|_{ij}$ in case i, j are both even. Since the last matrix has entries that commute with each other we can use (C.6) to find

$$|A|_{ij} = (-1)^{i+j} \frac{\det (X - \alpha Y^{-1}\beta)}{\det ([X - \alpha Y^{-1}\beta]^{ij})}. \tag{C.8}$$

Now for i, j both even $[X - \alpha Y^{-1}\beta]^{ij} = X^{ij} - \alpha^{i\emptyset} Y^{-1} \beta^{\emptyset j}$, where $\alpha^{i\emptyset}$ is the matrix obtained by deleting just the row i and $\beta^{\emptyset j}$ is obtained by deleting column j. So $\det ([X - \alpha Y^{-1}\beta]^{ij}) \det(Y^{-1}) = \mathrm{ber}\,(A^{ij})$ and the first part of the lemma follows by multiplying numerator and denominator of (C.8) by $\det (Y^{-1})$. For the second part use the decomposition

$$A = \begin{pmatrix} 1 & 0 \\ \beta X^{-1} & 1 \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y - \beta X^{-1}\alpha \end{pmatrix} \begin{pmatrix} 1 & X^{-1}\alpha \\ 0 & 1 \end{pmatrix}.$$

□

C.4. **Linear equations.** Now we think of the matrix $A = (a_{ij})_{i,j=1}^n$ as an even matrix over Λ. Then the quasideterminants $|A|_{ij}$, thought of as elements of Λ, can only exist if i, j have the same parity (since otherwise A^{ij} is never invertible). Even if i, j have the same parity $|A|_{ij}$ may or may not exist (as an element of Λ), but if it does then the quasideterminant $|A_i(y)|_{ij}$ exists also. (Recall that $A_i(y)$ is obtained by replacing the ith row of A by the row vector $y = (y_1, \ldots, y_n)$.) Indeed, according to (C.5), if $y = (y_1, \ldots, y_n)$, then, using that $A_i(y)^{ij} = A^{ij}$,

$$|A_i(y)|_{ij} = y_i - \sum_{\substack{p \neq i \\ pq \neq j}} y_p |A^{ij}|^{-1}_{qp} a_{qj}.$$

Hence, using Lemma C.3.1, we find, whenever $|A|_{ij}$ exists,

$$|A_i(y)|_{ij} = \begin{cases} (-1)^{i+j}\,\mathrm{ber}\,(A_i(y)) / \mathrm{ber}\,(A^{ij}) & \text{if } i \leq k, \\ (-1)^{i+j}\,\mathrm{ber}^*\,(A_i(y)) / \mathrm{ber}^*\,(A^{ij}) & \text{otherwise.} \end{cases} \tag{C.9}$$

Now consider the linear system

$$xA = y, \tag{C.10}$$

with x, y row vectors, and A an even matrix over Λ. Using the properties of the quasideterminant (C.10) implies

$$x_i|A|_{ij} = |A_i(y)|_{ij}.$$

Combining (C.9) and Lemma C.3.1 proves (C.4).

Of course, if one deals with infinite systems of equations (C.1) over a super commutative ring one has the same expressions (C.4) for the solutions, provided that one can define a Berezinian (with the usual properties) of the infinite matrices involved.

Appendix D. Calculation of a super tau function

The Baker functions for arbitrary line bundles over a super elliptic curve were computed in [Rab95b]. Note that a super elliptic curve, meaning an $N = 1$ super Riemann surface of genus one with *trivial* spin structure $\mathcal{N}$ is not a generic SKP curve and its Jacobian is not a supermanifold. Still, the corresponding super tau function can be computed using the Baker-tau relations (27,28) of [DS90]:

$$\text{(D.1)}\quad \operatorname{ber}[\tilde{B}(W,F,S^*)\cdot S^*] = \left\{\tau[W,F^T(1-Sz^{-1})^{-1}]/\tau(W,F^T)\right\}^{-1},$$

in the notation of that paper. We choose the 2×2 matrix S to be $\operatorname{diag}(\zeta,\zeta)$, with ζ an even variable; then the entries of the 2×2 matrix Baker function $\tilde{B}^{ij}(W,F,S^*)$ are simply the coefficients of the even Baker function $w_{\bar{0}} = B^{00}(\zeta)+B^{01}(\zeta)\theta$ and the odd Baker function $w_{\bar{1}} = B^{10}(\zeta)+B^{11}(\zeta)\theta$ for the subspace FW of $\mathcal{S}\mathrm{gr}$, where F is multiplication by some invertible formal Laurent series $f(z)+\phi(z)\theta$, viewed as a 2×2 matrix $\left[\begin{smallmatrix} f & 0\\ \phi & f\end{smallmatrix}\right]$. This action of F corresponds via the Krichever map to deforming $\mathcal{L}$ by tensoring it with a bundle whose transition function across the circle $|z| = 1$ is $f(z)+\phi(z)\theta$.

The results of [Rab95b] give the entries of the Baker matrix as:

$$\text{(D.2)}\qquad B^{00} = 1+\frac{\alpha\delta}{2\pi i}\left[\frac{\Theta'(a)\Theta'(\zeta-a)}{\Theta(a)\Theta(\zeta-a)}+\frac{\Theta'(a)^2}{\Theta(a)^2}\right],$$

$$\text{(D.3)}\qquad B^{01} = \alpha\frac{\Theta'(a)}{\Theta(a)} + \text{terms proportional to } \delta,$$

$$\text{(D.4)}\qquad B^{10} = \frac{\delta}{2\pi i}\left[\frac{\Theta'(\zeta-a)}{\Theta(\zeta-a)}+\frac{\Theta'(a)}{\Theta(a)}\right],$$

$$\text{(D.5)}\quad B^{11} = 1+\frac{\alpha\delta}{2\pi i}\left[\frac{\Theta''(a)}{\Theta(a)}-\frac{\Theta'(a)^2}{\Theta(a)^2}-\frac{\Theta''(\zeta-a)}{\Theta(\zeta-a)}+\frac{\Theta'(\zeta-a)^2}{\Theta(\zeta-a)^2}\right].$$

Here τ and δ are the even and odd moduli of the super elliptic curve X; all theta functions appearing are $\Theta_{11}(\bullet;\tau)$. The parameters (a,α)

label the point in the super Jacobian corresponding to the deformed bundle $\mathcal{L}$; if F is written in the form

$$F = \exp \sum_{i=1}^{\infty} (t_i z^{-i} + t_{i-\frac{1}{2}} \theta z^{-i}), \tag{D.6}$$

then they are linear combinations of the t_i plus constants labeling $\mathcal{L}$ itself.

Computing the superdeterminant, we find

$$\frac{\tau(W, F^T(1-\zeta z^{-1})^{-1})}{\tau(W, F^T)} = \frac{1 - \frac{\alpha\delta}{2\pi i}[\log \Theta(a-\zeta)]''}{1 - \frac{\alpha\delta}{2\pi i}[\log \Theta(a)]''}. \tag{D.7}$$

The function

$$(1-\zeta z^{-1})^{-1} = \exp \sum_{k=1}^{\infty} \frac{1}{k} \frac{\zeta^k}{z^k} \tag{D.8}$$

produces the usual shifts by ζ^k/k in the even parameters t_k. As a transition function for a line bundle, $\frac{z}{z-\zeta}$, it corresponds to the principal divisor $(0,0) - (\zeta, 0)$, so the corresponding deformation adds $-\zeta$ to the even coordinate a of the Jacobian, as we see on the right side of (D.7). We conclude that

$$\tau(W, F^T) = 1 - \frac{\alpha\delta}{2\pi i}[\log \Theta(a)]''. \tag{D.9}$$

Note, however, that if the 2×2 matrix F represents multiplication by $f(z) + \phi(z)\theta$ then F^T represents the action of $f(z) + \phi(z)\partial_\theta$, which in terms of the Krichever data is no longer just a deformation of $\mathcal{L}$, but of X as well. That is, the Baker function for the orbit of F enables us to calculate the tau function for the "dual" orbit of F^T, a totally different type of deformation.

We see also that, consistent with the discussion in subsection 4.3, the super tau function is a genuine function, not a section of a bundle, as the multivaluedness of the theta function is eliminated by the logarithmic derivatives. This is again due to the fact that the restricted group of F^T's considered here acts without central extension in the Berezinian bundle.

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