

MODEL ANSWERS TO THE NINTH HOMEWORK

1. Let

$$\{ X_\alpha \mid \alpha \in \Lambda \}$$

be a set of Hausdorff topological spaces and let

$$X = \prod_{\alpha \in \Lambda} X_\alpha$$

be their product. Suppose that x and y are distinct points of X . Then we may find $\alpha \in \Lambda$ such that

$$x_\alpha = p_\alpha(x) \neq p_\alpha(y) = y_\alpha.$$

As X_α is Hausdorff we may find disjoint open sets U_α and V_α , such that

$$x_\alpha \in U_\alpha \quad \text{and} \quad y_\alpha \in V_\alpha.$$

Let

$$U = p_\alpha^{-1}(U_\alpha) \quad \text{and} \quad V = p_\alpha^{-1}(V_\alpha).$$

Then U and V are disjoint open sets and

$$x \in U \quad \text{and} \quad y \in V.$$

Thus X is Hausdorff.

2. Let

$$\{ X_\alpha \mid \alpha \in \Lambda \}$$

be a set of connected topological spaces and let

$$X = \prod_{\alpha \in \Lambda} X_\alpha$$

be their product. If the product is empty then there is nothing to prove.

Otherwise we can find a point

$$z \in X.$$

We will refer to this as a base point. Note that if we take a product over a smaller indexing set, a subset of Λ , then the base point z gives rise to base points in these products.

Let $M \subset \Lambda$ be a finite set. Then

$$X_M = \prod_{\beta \in M} X_\beta$$

is connected, since it is the product of finitely many connected topological spaces. Using the base point z we get a base point in the product of the other topological spaces. This gives us a natural inclusion of X_M inside X

$$i_M: X_M \longrightarrow X$$

Given a point $x = (x_\beta) \in X_m$ send this to the point $y = (y_\alpha)$ with the same coordinates for each $\alpha \in M$ and with the coordinates of z otherwise:

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in M \\ z_\alpha & \text{otherwise.} \end{cases}$$

The image of i_M is connected, as i_M defines a homeomorphism with its image. Consider the union U of the images. The images all contain the base point z . Thus U is connected.

I claim that U is dense in X . It suffices to show that U intersects every basic open set. These are of the form

$$V = \prod_{\beta \in M} U_\beta \times \prod_{\alpha \in \Lambda \setminus M} X_\alpha$$

where $M \subset \Lambda$ is a finite set and $U_\beta \subset X_\beta$ is open. Note that the intersection of V with the image of X_M is non-empty, since it contains

$$i_M\left(\prod_{\beta \in M} X_\beta\right).$$

Thus the intersection of V with U is certainly non-empty.

Thus U is dense so that

$$X = \overline{U}.$$

But the closure of a connected set is connected. Thus X is connected.

3. (i) Let $\sim$ be the relation,

$$x \sim y \quad \text{if and only if} \quad f(x) = f(y).$$

It is not hard to see that $\sim$ is an equivalence relation and that the equivalence classes are the fibres of f . It is also clear that $\sim$ is the only equivalence relation with this property.

By the universal property of the quotient space there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y. \\ \downarrow q & \nearrow g & \\ X/\sim & & \end{array}$$

It is not hard to see that g is a bijection. It is by assumption continuous.

If f satisfies the universal property of a quotient for some equivalence relation then it must be for this equivalence relation. In this case g is a homeomorphism. We already saw that $X/\sim$ has the finest topology that makes q continuous. The same is therefore true of f . Therefore f is a quotient map.

Now suppose $V \subset Y$ is open if and only if $f^{-1}(V)$ is open. Note that q has a similar property, $W \subset X/\sim$ is open if and only if $q^{-1}(W)$ is open. But then h induces a bijection between the open sets of Y and $X/\sim$. Thus h is a homeomorphism.

(ii) Suppose that f is open. Let $V \subset Y$ be a subset and let $U = f^{-1}(V)$. If V is open then U is open, as f is continuous. Now suppose that U is open. Then $f(U)$ is open, as f is open. But $f(U) = V$ and so V is open. But then f is a quotient map.

(iii) Note first that f is a quotient map if and only if for every subset $G \subset Y$, G is closed if and only if $F = f^{-1}(G)$ is closed. Indeed if we put $V = Y \setminus G$ and $U = X \setminus F$ then note that $U = f^{-1}(V)$.

Suppose that $G \subset Y$. If G is closed then $F = f^{-1}(G)$ is closed, as f is continuous. Now suppose that F is closed. Then $f(F)$ is closed, as f is closed. But $G = f(F)$ and so G is closed. But then f is a quotient map.

(iv) Let $F \subset X$ be closed. Then F is compact as X is compact. Let $G = f(F)$. Then G is compact. As Y is Hausdorff, it follows that G is closed. But then f is closed and so f is a quotient map.

(v) Let V be a subset of Y . Let $U = f^{-1}(V)$ and suppose that U is open. We have

$$\begin{aligned} g^{-1}(U) &= g^{-1}(f^{-1}(V)) \\ &= (f \circ g)^{-1}(V) \\ &= V. \end{aligned}$$

As U is open and g is continuous, it follows that V is open. But then f is a quotient map.

(vi) There are two ways to show this. Note that p is continuous and surjective.

Consider an open set of the form $U \times V$, where U is open in X and V is open in Y . Then

$$p(U \times V) = U,$$

which is open. As sets of the form $U \times V$ are a basis for the topology, it follows that p is open. Thus p is a quotient map.

Aliter: Pick a base point y_0 of X . Define a function

$$g: X \longrightarrow X \times Y$$

by the rule

$$g(x) = (x, y_0).$$

Then g is continuous, as its components are continuous functions (the identity and a constant function). Consider the composition $p \circ g$. We have

$$\begin{aligned}(p \circ g)(x) &= p(g(x)) \\ &= p(x, y_0) \\ &= x.\end{aligned}$$

Thus $p \circ g$ is the identity. Thus p has a right inverse and so p is a quotient map.

(vii) Let

$$i: Y \longrightarrow X$$

be the natural inclusion. Then i is continuous. Consider the composition $r \circ i$. We have

$$\begin{aligned}(r \circ i)(y) &= r(i(y)) \\ &= p(y) \\ &= y.\end{aligned}$$

Thus $r \circ i$ is the identity and r has a right inverse. Thus r is a quotient map.

(viii) Let

$$f: X \longrightarrow Y \quad \text{and} \quad g: Y \longrightarrow Z$$

be two quotient maps. Consider the composition

$$h = g \circ f: X \longrightarrow Z.$$

h is continuous and surjective, since both f and g are continuous and surjective. Suppose that $W \subset Z$ and $U = h^{-1}(W)$ is open. Let $V = g^{-1}(W)$. Then

$$\begin{aligned}f^{-1}(V) &= f^{-1}(g^{-1}(W)) \\ &= (g \circ f)^{-1}(W) \\ &= h^{-1}(W) \\ &= U.\end{aligned}$$

As f is a quotient map and U is open, it follows that V is open. As g is a quotient map and V is open, it follows that W is open.

Thus h is a quotient map.

4. f is certainly continuous and surjective. Let

$$g: \mathbb{R} \longrightarrow X$$

be the function

$$g(x) = (x, 0).$$

Then g is continuous. It is easy to see that g is a right inverse of f . Thus f is a quotient map.

Let

$$F = \{ (x, y) \mid xy = 1 \} \subset X.$$

Then F is closed. Indeed, it is closed in $\mathbb{R}^2$. But the image of F is

$$(0, \infty)$$

which is not closed in $\mathbb{R}$. Thus p is not closed.

Now consider

$$\{ (x, y) \mid x \geq 0, y \neq 0 \} \subset X.$$

Then this is open in X (for example it is the intersection of the open set

$$\{ (x, y) \mid x > -1, y \neq 0 \} \subset \mathbb{R}^2$$

with X). But the image is

$$[0, \infty),$$

which is not open in $\mathbb{R}$. Thus p is not open.

5. We want two subspaces of $\mathbb{RP}^2$ that are homeomorphic to the closed unit disc and to the Möbius strip which only intersect along their boundary circles.

The simplest way to think about $\mathbb{RP}^2$ is that there is a double cover

$$f: S^2 \longrightarrow \mathbb{RP}^2$$

which identifies antipodal points. We informally think of the globe as a sphere. Consider everything Y_n above the tropic of Cancer, in the upper hemisphere,

$$\{ (x, y, z) \mid z \geq a \} \cap S^2,$$

for some appropriate $a \in (0, 1)$. Y_n is closed and it is homeomorphic to the closed unit disc, via projection

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^2 \quad \text{given by} \quad (x, y, z) \longrightarrow (x, y)$$

down to the plane $z = 0$. If we take the image of Y_n in $\mathbb{RP}^2$ and its inverse image Y_1 in S^2 , then we get everything above the tropic of Cancer and everything below the tropic of Capricorn, $Y_1 = Y_n \amalg Y_s$. Y_1 is closed, as it is the disjoint union of two closed discs, and the quotient map identifies antipodal points. Thus the image X_1 of Y_1 in $\mathbb{RP}^2$ is homeomorphic to a closed disc.

The closed set Y between the tropics is homeomorphic to a cylinder. Indeed Y is the intersection of the sphere with the intersection of the two closed half spaces

$$\{(x, y, z) \mid z \leq a\} \quad \text{and} \quad \{(x, y, z) \mid z \geq -a\}$$

If we project from the origin onto the set $C \times \mathbb{R}$, where C is a circle of radius 2 centred at the origin, then the image Y_2 of Y is a cylinder, since it will be the subset

$$Y_2 = S^1 \times [-2a, 2a] \subset S^1 \times \mathbb{R}.$$

Note also that the equivalence relation induced on Y_2 by the equivalence relation on Y has not changed (we identify the vector v with the vector $-v$), since the identification of Y with Y_2 is given by projection from the origin.

If we take the image X_2 of Y_2 in $\mathbb{RP}^2$ then the inverse image of X_2 is Y_2 . Thus Y_2 maps two to one to X_2 . It is easy to see that the two circles on Y_2 gets mapped to one circle on X_2 . Thus X_1 and X_2 only intersect along this circle.

$Y_1 \amalg Y_2$ is compact and $\mathbb{RP}^2$ is Hausdorff. Thus the induced continuous map

$$Y_1 \amalg Y_2 \longrightarrow \mathbb{RP}^2$$

is therefore a quotient map. By the universal property of the quotient map

$$S^2 \longrightarrow \mathbb{RP}^2$$

we get a continuous and surjective map

$$X_1 \amalg X_2 \longrightarrow \mathbb{RP}^2$$

which again has to be a quotient map. This quotient map simply identifies the two boundary circles of X_1 and X_2 .

It remains to show that X_2 is homeomorphic to a Möbius strip. Imagine cutting Y_2 along two antipodal vertical line segments L_1 and L_2 to get two closed rectangles R_1 and R_2 . Say the segments from $(0, 2, -2a)$ to $(0, 2, 2a)$ and $(0, -2, -2a)$ to $(0, -2, 2a)$. The equivalence relation identifies antipodal points of R_1 with R_2 . X_2 is the quotient of

$$\pi: R_1 \amalg R_2 \longrightarrow X_2$$

via the equivalence relation which identifies v and $-v$ and then the points of the boundary corresponding to L_1 and L_2 . Let R be the quotient we get by just identifying v and $-v$,

$$R_1 \amalg R_2 \longrightarrow R$$

It is easy to see that R is a copy of R_1 , so that R is homeomorphic to $[0, 1]^2$. Moreover by the universal property of π there is a continuous map

$$R \longrightarrow X_2$$

which identifies two opposite line segments. This map is a quotient map, since $[0, 1]^2$ is compact and X_2 is Hausdorff. On the other hand, it is not hard to check (best using a couple of diagrams) that we get to identify the two opposite sides of R with the opposite orientation. This is the standard way to construct the Möbius strip. Thus X_2 is homeomorphic to the Möbius strip.

Challenge Problems: (Just for fun)

6. First note that all three underlying manifolds, S^n , $S^1 \times S^1$ and $\mathbb{RP}^2$ are compact. It is therefore clear that $\lambda(M) > 1$, since if we can a topological space X with one open subset U , then $X = U$ and $\mathbb{R}^n$ is not compact.

We already saw in Homework 5 that if you take S^n and delete a point then you get something homeomorphic to $\mathbb{R}^n$. If N and S denote the north and south pole, and we put $U_n = S^n \setminus \{N\}$ and $U_s = S^n \setminus \{S\}$ then U_n and U_s are both homeomorphic to $\mathbb{R}^n$ and their union is the whole of S^n . Thus

$$\lambda(S^1) = \lambda(S^2) = \lambda(S^n) = 2.$$

Now note that if M^m and N^n are two manifolds then

$$\lambda(M \times N) \leq \lambda(M) \times \lambda(N).$$

Indeed if

$$\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\} \quad \text{and} \quad \mathcal{V} = \{V_\beta \mid \beta \in M\}$$

are open covers of M and N then

$$\mathcal{W} = \{W_{\alpha,\beta} = U_\alpha \times V_\beta \mid (\alpha, \beta) \in \Lambda \times M\}$$

is an open cover of $M \times N$. But if U_α is homeomorphic to $\mathbb{R}^m$ and V_β is homeomorphic to $\mathbb{R}^n$ then $W_{\alpha,\beta}$ is homeomorphic to $\mathbb{R}^{m+n}$.

Thus

$$2 \geq \lambda(S^1 \times S^1) \leq 4.$$

Can we do better than this? TBC.