

MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. As X is Hausdorff and A is compact it follows that A is closed. Similarly B is closed. It follows that $A \cap B$ is closed in A . As A is compact, it follows that $A \cap B$ is compact.
2. (i) Pick $(x, y) \notin \Gamma_f$. Then $y \neq f(x)$. As Y is Hausdorff we may find V and W open and disjoint in Y such that $f(x) \in V$ and $y \in W$. Let $U = f^{-1}(V)$. Then $x \in U$ and U is open. Consider $U \times W$. As $f(U) = V$ is disjoint from W it follows that

$$(x, y) \in U \times W \subset (X \times Y) \setminus \Gamma_f.$$

As $U \times W$ is open it follows that $(X \times Y) \setminus \Gamma_f$ is open. But then Γ_f is closed.

Aliter: Consider the function

$$\phi: X \times Y \longrightarrow Y \times Y \quad \text{given by} \quad \phi(x, y) = (f(x), y).$$

It is easy to check that ϕ is continuous and that

$$\Gamma_f = (\phi)^{-1}(\Delta)$$

where $\Delta \subset Y \times Y$ is the diagonal. As Y is Hausdorff, the diagonal is closed and as ϕ is continuous it follows that $\Gamma_f \subset X \times Y$ is closed.

- (ii) Let $F \subset X \times Y$ be a closed set. Let

$$W = (X \times Y) \setminus F.$$

Then W is open. Consider the image $p(F)$. Pick a point $x \notin p(F)$. Then

$$p^{-1}(x) = \{x\} \times Y$$

is contained in W . As Y is compact, the tube lemma implies that we may find $U \subset X$ such that

$$p^{-1}(U) = U \times Y \subset W.$$

But then

$$x \in U \subset X \setminus p(F).$$

It follows that $X \setminus p(F)$ is open, so that $p(F)$ is closed. Thus p is closed.

- (iii) The natural inclusion of

$$i: \Gamma_f \longrightarrow X \times Y$$

followed by the projection p onto X is a continuous function,

$$g: \Gamma_f \longrightarrow X,$$

since it is the composition of continuous functions. As Γ_f is closed in $X \times Y$, it follows that i is closed. As p is closed and the composition of closed maps is closed (easy check) it follows that g is closed.

The two functions

$$i_X: X \longrightarrow X \quad \text{and} \quad f: X \longrightarrow Y$$

give a function

$$X \longrightarrow X \times Y \quad \text{given by} \quad x \longrightarrow (x, f(x)),$$

by the universal property of the product. It is clear that the image is Γ_f so that we get a function

$$h: X \longrightarrow \Gamma_f.$$

h is the inverse of g . Thus g is a bijection and so g is a homeomorphism. But then h is a homeomorphism. In particular h is continuous. The composition of h and the projection $q: X \times Y \longrightarrow Y$ is f . As q is continuous it follows that f is continuous.

3. Pick $a \in A$. As $\{a\} \times B \subset W$ we may find open sets

$$\{U_\alpha \times V_\alpha \mid \alpha \in \Lambda\}$$

such that

$$\{a\} \times B \subset \bigcup_{\alpha \in \Lambda} U_\alpha \times V_\alpha \subset W.$$

As B is compact, the tube lemma implies that we may find $a \in U_a$ open in X and V_a open in Y such that

$$\{a\} \times B \subset U_a \times V_a \subset W.$$

As

$$\mathcal{U} = \{U_a \cap A \mid a \in A\}$$

is an open cover of A and A is compact we may find finitely many open sets $U_1, U_2, \dots, U_k$ and $V_1, V_2, \dots, V_k$ of X and Y such that

$$A \subset \bigcup U_i = U \quad \text{and} \quad B \subset \bigcap V_i = V.$$

U is open as it is the union of open sets and V is open as it is the finite intersection of open sets. In this case

$$A \times B \subset U \times V \subset W.$$

4. (i) If the image of h lands in the diagonal then $h(x) = (f(x), g(x)) \in \Delta$ and $f(x) = g(x)$ for all $x \in X$. But then $f = g$.

(ii) By assumption the restriction of h to A lands inside the diagonal. It follows that the image of A lands inside the diagonal:

$$h(A) \subset \Delta.$$

On the other hand, Δ is closed as Y is Hausdorff. We have

$$\begin{aligned} h(X) &= h(\overline{A}) \\ &\subset \overline{h(A)} \\ &\subset \overline{\Delta} \\ &= \Delta. \end{aligned}$$

But then $f = g$.

5. (i) Let

$$W = X \times Y \setminus \Delta.$$

As X is Hausdorff, it follows that Δ is closed so that W is open. We have

$$A \times B \subset W.$$

By (3) we may find U and V open so that

$$A \times B \subset U \times V \subset W.$$

It follows that U and V are disjoint open subsets, $A \subset U$ and $B \subset V$.

(ii) The arbitrary intersection of closed sets is closed. Thus

$$F = \bigcap_{\alpha} F_{\alpha}$$

is certainly closed.

Suppose that F is a disjoint union of two open sets A and B . Then A and B are closed in F . As F is closed, A and B are closed. As X is compact, it follows that A and B are compact. By (i) we may find disjoint open subsets U and V containing A and B . But then $U \cap F = A$ and $V \cap F = B$.

Consider

$$G_{\alpha} = F_{\alpha} \setminus (U \cup V).$$

This is a closed set. It is non-empty as F_{α} is connected.

Consider the collection

$$\{ G_{\alpha} \mid \alpha \in \Lambda \}.$$

These are totally ordered by inclusion. Thus any finite collection can be ordered, from smallest to largest. In particular if G_{α} is non-empty for all α then any finite collection of sets has non-empty intersection. As X is compact it follows that the intersection of every element is non-empty,

$$G = \bigcap_{\alpha} G_{\alpha} \neq \emptyset.$$

But then

$$F \setminus (U \cup V) = G$$

is non-empty, a contradiction. Thus F is connected.

Challenge Problems: (Just for fun)

6. We have already seen that if Y is compact then p is closed.

Now suppose that p is always closed. Let

$$\mathcal{V} = \{ V_\alpha \mid \alpha \in \Lambda \}$$

be an open cover of Y . Let X be the set you get by adjoining a new element ω . Put a topology on X as follows: $F \subset X$ is closed if and only if either

$$\omega \in F \quad \text{or} \quad F \subset \bigcup_{\beta \in M} V_\beta,$$

where $M \subset \Lambda$ is finite.

We check that this is a topology on X . $\emptyset$ is closed, since it is a subset of any union, and X is closed as it contains ω .

Suppose we intersect a collection of closed sets. If one of them is a subset of the union of finitely many elements of $\mathcal{V}$ then the intersection is also a subset. Otherwise every one of our closed sets contains ω and so the intersection contains ω . Thus the arbitrary intersection of closed sets is closed.

Now consider taking the union of two closed sets. If one of them contains ω then so does the union. Otherwise both closed sets are contained in the union of finitely many elements of $\mathcal{V}$. But then the union is contained in the union of these two finite subsets of $\mathcal{V}$. Thus the union of two closed sets is closed.

It follows that we have a topology. Let Δ be the diagonal in $Y \times Y$. Consider the natural inclusion

$$Y \times Y \longrightarrow X \times Y$$

Let D be the closure of the image of Δ inside $X \times Y$.

Pick $y \in Y$. Then $y \in V_\alpha$, some $\alpha \in \Lambda$. Now $V_\alpha \subset X$ is closed. Thus $U = X \setminus V_\alpha$ is open in X . Consider the open set

$$(\omega, y) \in U \times V_\alpha \subset X \times Y.$$

Note that this set does not intersect the image of Δ . It follows that (ω, y) does not belong to D .

As y was arbitrary, it follows that D is equal to the image of Δ . As p is closed,

$$Y = p(D) \subset X$$

is closed. As $\omega \notin Y$ it follows that Y is the finite union of elements of $\mathcal{U}$, that is, $\mathcal{U}$ has a finite subcover. But then Y is compact.