

## MODEL ANSWERS TO THE SEVENTH HOMEWORK

1. If  $X$  is empty there is nothing to prove. Otherwise, suppose we have a cover

$$\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\}$$

by open sets. As  $X$  is not empty,  $\Lambda$  is not empty and so we may find  $U_\alpha \in \mathcal{U}$ , non-empty. Let

$$F = X \setminus U_\alpha.$$

By assumption,  $F$  is finite. For every  $f$  element of  $F$ , we may pick  $\alpha_f$  such that

$$f \in U_{\alpha_f}.$$

In this case

$$\{U_{\alpha_f} \mid f \in F\} \cup U_\alpha$$

is a finite subcover. But then  $X$  is compact.

2. Let

$$\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\}$$

be the complements of the closed sets  $F_\alpha$ . Then  $U_\alpha$  are open.

Suppose that  $M$  is a finite subset of  $\Lambda$ . By assumption

$$\mathcal{F}_0 = \{F_\beta \mid \beta \in M\}$$

has non-empty intersection. Let's suppose that  $a$  belongs to the intersection. It follows that  $a$  does not belong to the union of

$$\mathcal{U}_0 = \{U_\beta \mid \beta \in M\}.$$

Thus  $\mathcal{U}_0$  is not a finite subcover. Thus  $\mathcal{U}$  has no finite subcover. As  $X$  is compact, the only possibility is that  $\mathcal{U}$  is not a cover. Thus we may find  $a \in X$  that does not belong to any  $U_\alpha$ . But then  $a$  belongs to every  $F_\alpha$ , so that  $a$  belongs to the intersection of the elements of  $\mathcal{F}$ . But then the intersection of the elements of  $\mathcal{F}$  is non-empty.

3. True or false? If true then give a proof and if false then give a counterexample.

(i) True. The image of compact is compact. Thus  $Y = f(X)$  is compact. Thus  $Y$  is closed and bounded. Let  $s$  be the supremum of  $Y$ . As  $Y$  is bounded,  $s \in \mathbb{R}$ . As  $Y$  is closed,  $Y$  contains all of its accumulation points. Thus  $s \in Y$ . Similarly  $Y$  contains its infimum  $i$ . Thus  $f$  achieves its maximum and minimum value.

(ii) True. Pick  $x_1 \neq x_2 \in X$ . Let

$$y_i = f(x_i) \quad \text{for} \quad i = 1, 2.$$

As  $f$  is injective, it follows that  $y_1 \neq y_2$ . As  $Y$  is Hausdorff, we may find

$$y_i \in V_i \quad \text{for} \quad i = 1, 2.$$

where  $V_1$  and  $V_2$  are open and disjoint. Let

$$U_i = f^{-1}(V_i) \quad \text{for} \quad i = 1, 2.$$

Then

$$x_i \in U_i \quad \text{for} \quad i = 1, 2.$$

$U_1$  and  $U_2$  are open as  $f$  is continuous and they are disjoint as  $V_1$  and  $V_2$  are disjoint.

(iii) True. Let

$$\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\}$$

be an open cover of  $A \cup B$ . Then we get open covers

$$\mathcal{U}_A = \{U_\alpha \cap A \mid \alpha \in \Lambda\}$$

of  $A$  and of  $B$ :

$$\mathcal{U}_B = \{U_\alpha \cap B \mid \alpha \in \Lambda\}.$$

As  $A$  and  $B$  are compact, it follows that both of these have finite open subcovers.

$$\{U_\beta \cap A \mid \beta \in M_A\} \quad \text{and} \quad \{U_\beta \cap B \mid \beta \in M_B\}.$$

In this case

$$\{U_\beta \mid \beta \in M_A \cup M_B\}$$

is finite subcover of  $\mathcal{U}$ . Hence  $A \cup B$  is compact.

(iv) False. Let

$$X = \mathbb{N} \cup \{\omega_1, \omega_2\}.$$

Consider the following topology on  $X$ . A subset  $U$  is open if, either

$$\begin{cases} U \subset \mathbb{N} & \text{or} \\ \mathbb{N} \subset U. \end{cases}$$

It is not hard to check this does define a topology. Let

$$A = \mathbb{N} \cup \{\omega_1\} \quad \text{and} \quad B = \mathbb{N} \cup \{\omega_2\}.$$

Note that the only open subset of  $A$  containing  $\omega_1$  is the whole of  $A$ . Thus  $A$  is compact. Similarly  $B$  is compact. But  $A \cap B = \mathbb{N}$ , with the discrete topology, which is not compact.

4. Note that  $Y$  is compact as  $f$  is surjective (and  $X$  is Hausdorff as  $f$  is injective; we won't need this).

Let  $F \subset X$  be closed. Then  $F$  is compact as  $X$  is compact. The image of  $G = f(F)$  is compact as  $f$  is continuous. As  $Y$  is compact and Hausdorff, it follows that  $G$  is closed. Thus  $f$  is **closed**, meaning

that the image of closed is closed. As  $f$  is a bijection, it follows that  $f$  is **open**, meaning that the image of open is open (NB: if  $f$  is not a bijection then there is not necessarily any connection between  $f$  being open and  $f$  being closed). But then  $f$  induces a bijection between the open sets in  $X$  and  $Y$ , so that  $f$  is a homeomorphism.

### Challenge Problems: (Just for fun)

5. Let  $T$  be the topological space determined by the letter  $T$ . Suppose we are given an uncountable collection

$$\{\mathcal{T}_\alpha \mid \alpha \in \Lambda\}$$

of subset of  $\mathbb{R}^2$ , homeomorphic to  $T$ . Points of  $T$  come in three types. The removal of most points leaves two connected components. There is one point whose removal leaves three connected components. Let us call this the centre. There are three points which are not cut points. Let us call these endpoints. Consider a component  $I$  of  $T - \{c\}$ , where  $c$  is the centre. Then  $I$  is homeomorphic to an open interval. The closure of  $I$  contains two more points, the centre and an endpoint, and it is homeomorphic to a closed interval. Let call this a segment. Given  $\mathcal{T}_\alpha$  pick a circle that intersects all three segments connected to the centre. We may assume that the centre of  $\mathcal{T}_\alpha$  belongs to the interior of the open disk determined by the circle. Perturbing the circle, we may assume that the centre of the circle has rational coordinates. Shrinking the radius of the circle, and using the fact that a segment is compact, we may assume that the circle intersects each segment only once. Shrinking the radius of the circle we may assume that the radius is a rational number.

As  $\Lambda$  is uncountable and there are only countably many circles whose centre has rational coordinates and whose radius is rational, we may assume that there is one circle that works for all  $\mathcal{T}_\alpha$ . Consider the closed ball determined by the circle. The removal of the circle plus  $\mathcal{T}_\alpha$  gives an open set  $U_\alpha$ .  $U_\alpha$  has three connected components (equal to the path components), by the Jordan curve theorem (I don't see a way not to appeal to this). Pick a rational point from each such component.

As there are countably many such choices, we may assume that we pick the same points for each  $\mathcal{T}_\alpha$ . But this is not possible. Suppose that  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are two such. The centre  $c_2$  of  $\mathcal{T}_2$  must belong to one of the three path components determined of  $U_1$ . Pick an endpoint of  $\mathcal{T}_2$  that does not belong to the closure of this component. The segment from the centre of  $\mathcal{T}_2$  to this endpoint gives a path from one path component of  $U_1$  to another, which is not possible.

We show that we can find uncountably many copies of  $T$  in  $\mathbb{R}^n$ , provided  $n \geq 3$ . It is enough to do this for  $n = 3$ .

Let  $T \subset \mathbb{R}^2$  be a standard copy of  $T$ . Then

$$\{T \times \{c\} \mid c \in \mathbb{R}\}$$

is a set of uncountably many disjoint copies of  $T$ .

Thus  $\mathbb{R}^2$  and  $\mathbb{R}^n$  are not homeomorphic if  $n > 2$ . We already know that if  $\mathbb{R}^1$  and  $\mathbb{R}^n$  are homeomorphic then  $n = 1$ . Thus if  $\mathbb{R}^2$  and  $\mathbb{R}^n$  are homeomorphic then  $n = 2$ .

6. We say that a topological space  $X$  is **Noetherian** if the set of closed subsets satisfies the DCC (descending chain condition). This means that every infinite decreasing sequence of closed sets

$$F_1 \supset F_2 \supset \cdots \supset F_n \dots$$

eventually stabilises, that is, there is an index  $k$  such that if  $i$  and  $j > k$  then  $F_i = F_j$ .

(i) Let

$$F_1 \supset F_2 \supset \cdots \supset F_n \dots$$

be a decreasing sequence of closed sets. If  $F_i = X$  for all  $i$ , there is nothing to prove. Otherwise there is no harm in assuming that  $F_1 \neq X$ , in which case  $F_1$  is finite. Pick  $k$  so that  $F_k$  has the smallest number of elements. Then  $F_i = F_j$  for all  $i$  and  $j \geq k$ .

(ii) Let

$$\{U_\alpha \mid \alpha \in \Lambda\}$$

be an open cover. We define a sequence of open sets

$$U_1 \subset U_2 \subset \cdots \subset U_n \dots$$

Let  $U_0 = \emptyset$ . Suppose we have defined  $U_i$ . If  $U_i = X$  then put  $U_{i+1} = X$ . Otherwise pick  $x \notin U_i$  and pick  $\alpha$  such that  $x \in U_\alpha$ . Let

$$U_{i+1} = U_i \cup U_\alpha.$$

The increasing sequence of open sets gives rise to a decreasing sequence of closed sets, simply by taking complements. This sequence of closed sets must eventually stabilise. Thus the sequence of open sets must eventually stabilise. By construction this only happens if  $X = U_i$ , some  $i$ . As  $U_i$  is the union of  $i$  elements of the open cover, this means we have found a finite subcover.

(iii) Closed subsets of  $k^n$  are the same as subvarieties. These are in correspondence with ideals. This correspondence is inclusion reversing, if

$$W \subset V \quad \text{if and only if} \quad I(V) \subset I(W).$$

As  $k[x_1, x_2, \dots, x_n]$  is a Noetherian ring, it follows that the set of ideals satisfies the ACC. But then the set of subvarieties satisfies the DCC. It follows that  $k^n$  is Noetherian.

(iv) Suppose that  $P(X)$  does not hold. We are going to define a strictly decreasing sequence of closed subsets of  $X$ ,

$$X_1 \supset X_2 \supset \dots \supset X_n \dots$$

with the property that  $P(X_i)$  does not hold, for all  $i$ .

Let  $X_1 = X$ . Suppose that we have constructed  $X_1, X_2, \dots, X_n$ . By assumption  $P(X_n)$  does not hold. Thus we may find a proper closed subset  $Y \subset X_n$  such that  $P(Y)$  does not hold. Let  $X_{n+1} = Y$ .

This completes the construction of  $X_1, X_2, \dots$ . But this contradicts the fact that  $X$  is Noetherian.

Thus  $P(X)$  must hold for all Noetherian topological spaces.

(v) Let  $P$  denote the property that a topological space decomposes as a finite union of closed irreducible subsets.

Let  $X$  be a Noetherian topological space. If  $X$  is irreducible then  $P(X)$  holds. Otherwise we may find two proper closed subsets  $Y_1$  and  $Y_2$  such that  $X = Y_1 \cup Y_2$ . By Noetherian induction  $Y_1$  and  $Y_2$  are each the union of finitely many closed irreducible subsets,

$$Y_1 = \bigcup_i F_i \quad \text{and} \quad Y_2 = \bigcup_j G_j.$$

But then

$$\begin{aligned} X &= Y_1 \cup Y_2 \\ &= \bigcup_i F_i \cup \bigcup_j G_j. \end{aligned}$$

$F_1, F_2, \dots, F_k$  are closed subsets of  $X$  as  $Y_1$  is closed. Similarly  $G_1, G_2, \dots, G_l$  are closed subsets of  $X$ .

Thus  $P(X)$  holds.

By Noetherian induction it follows that every Noetherian topological space is a finite union of closed irreducible subsets.

Now suppose there were two ways to decompose  $X$ ,

$$F_1 \cup F_2 \cup \dots \cup F_k = G_1 \cup G_2 \cup \dots \cup G_l$$

as the finite union of closed irreducible subsets, which are both minimal. Consider  $F_1$ . We may decompose this into the union of finitely many closed subsets

$$F_1 = \bigcup_j (F_1 \cap G_j).$$

As  $F_1$  is irreducible, it follows that we may find  $j$  such that

$$F_1 \subset G_j.$$

Similarly we may find  $i$  such that

$$G_j \subset F_i.$$

As the decomposition on the LHS is minimal and  $F_1 \subset F_i$ , it follows that  $i = 1$ . But then

$$F_1 = G_1.$$

It is easy conclude from this that the two decompositions are the same, up to reordering.