

MODEL ANSWERS TO THE SIXTH HOMEWORK

1. X is connected if and only if the cardinality of X is infinite or at most one.

If X has cardinality at most one then X is obviously connected. If the cardinality of X is finite but at least two then X is disconnected, since it has the discrete topology. Any partition of X is a partition into open sets.

Suppose that X is infinite and $X = F \cup G$, where F and G are closed and disjoint. One of F and G must be infinite. Suppose that F is infinite. Then $F = X$ and so X is connected.

2. Let

$$f: Z \longrightarrow \{0, 1\}$$

be a continuous function. Then $f|_Y$ is continuous. As Y is connected, f is constant. Suppose that $z \in Z$ and let

$$U = f^{-1}(f(z)).$$

Then U is open. As Y is dense in Z we may find $y \in Y \cap U$. In this case

$$f(z) = f(y).$$

Thus f is constant and so Z is connected.

3. Let $Y \subset \mathbb{R}^n$ be a subset.

(i) Let p and q be two points of Y . Define a path

$$f: [0, 1] \longrightarrow Y$$

by the rule

$$f(t) = (1 - t)p + tq.$$

Then f is a path from p to q . Thus Y is path-connected.

(ii) Suppose that Y is star-shaped about y_0 . Arguing as in (i) there is a path from y_0 to every point of Y . But then Y is path-connected, as the equivalence class generated by y_0 is the whole of Y .

4. (i) False. For example the continuous image of a connected set is connected. But the image of $\mathbb{Q}^2$ under projection down to the first factor is the set of rationals. This is not connected, as it is not an interval.

(ii) False. Path-connected implies connected but we saw in (i) that this is not even connected.

(iii) True (somewhat surprisingly). Consider a line of the form

$$l = \{ (x, y) \mid ax + by = c \}$$

where a and $b \in \mathbb{Z}$ and $c \in \mathbb{I}$. Note that l is completely contained in

$$\mathbb{R}^2 \setminus \mathbb{Q}^2.$$

Indeed, if $(\alpha, \beta) \in \mathbb{Q}^2$ then

$$a\alpha + b\beta \in \mathbb{Q}.$$

But then $(\alpha, \beta) \notin l$, as c is irrational.

Note that the condition that a and b are integers is the same as to say that the slope of l is rational (or l is vertical). In this case the condition that c is irrational is equivalent to the condition that it passes through a point with exactly one rational coordinate.

Let p and q be two points of

$$\mathbb{R}^2 \setminus \mathbb{Q}^2$$

Suppose that $p \notin \mathbb{I}^2$ (so that one coordinate of p is rational and the other is irrational). Pick any line l through q of the given form (for example, a horizontal or vertical line). Then there is a path from q to any point r of this line. Now pick a line m through p whose slope is not the same as the line. Then m and l intersect, as their slopes are different. It follows that there is a path from p to q .

(iv) True, since path-connected implies connected.

(v) True. Let y_1 and y_2 be two points of Y . Pick points x_0 and x_1 of X such that

$$f(x_i) = y_i \quad \text{for} \quad i = 0, 1.$$

Let

$$g: [0, 1] \longrightarrow X$$

be a path from x_0 to x_1 . Then g is a continuous function and

$$g(i) = x_i.$$

Let

$$h = f \circ g: [0, 1] \longrightarrow X$$

be the composition. Then h is continuous as it is the composition of continuous functions and

$$\begin{aligned} h(i) &= f(g(i)) \\ &= f(x_i) \\ &= y_i \end{aligned}$$

Thus h is a path from y_1 to y_2 and so Y is path-connected.

(vi) True. Suppose that

$$(x_i, y_i) \in X \times Y \quad \text{for} \quad i = 0, 1.$$

By assumption there are paths

$$f: [0, 1] \longrightarrow X \quad \text{and} \quad g: [0, 1] \longrightarrow Y$$

from x_0 to x_1 and from y_0 to y_1 , so that $f(i) = x_i$ and $g(i) = y_i$, $i = 0$ and 1 . Consider the function

$$h: [0, 1] \longrightarrow X \times Y$$

given by

$$h(t) = (f(t), g(t)).$$

Then $f = p \circ h$ and $g = q \circ h$ are continuous, so that h is continuous. But

$$h(i) = (x_i, y_i) \quad \text{for} \quad i = 0, 1,$$

so that h is a path from (x_0, y_0) to (x_1, y_1) . But then $X \times Y$ is path-connected.

(vii) False. One of the path components of $\overline{B} = A \cup B$, where B is the graph of the sine curve, is B itself. This is not closed.

(viii) False. For example, the rationals are everywhere disconnected, so that the connected components are points. But these are not open.

5. (i) The point of tangency is not a cut point.

We may assume that the point of tangency is the origin and the centres of the two circles are on the x -axis (first translate so that the point of tangency is the origin, then rotate so that the centre of one circle lies on the x -axis. It follows that the other centre will lie on the x -axis. These two maps are easily seen to be homeomorphisms). In this case the line of tangency is the y -axis.

If we remove the point of tangency then consider the two open sets

$$U = (-\infty, 0) \times \mathbb{R} \quad \text{and} \quad V = (0, \infty) \times \mathbb{R}.$$

They are disjoint and the two open sets $U \cap Y$ and $V \cap Y$ cover Y . Thus Y is disconnected.

Now suppose we remove any other point. Then, up to homeomorphism, we have a circle and an open interval, intersecting at a point. This is path-connected and path-connected implies connected.

(ii) If we restrict f to $X - \{x\}$ then we get an injective function to Y . This factors through the natural inclusion of $Y - \{y\}$ in Y . Thus we get a continuous map from $X - \{x\}$ to $Y - \{y\}$. This map is a bijection and a similar argument shows that the inverse map is continuous. Thus $X - \{x\}$ and $Y - \{y\}$ are homeomorphic. But then

$X - \{x\}$ is connected if and only if $Y - \{y\}$ is connected and so x is a cut point if and only if y is a cut point.

(iii) Every point of $(0, 1)$ is a cut point. If $a \in (0, 1)$ then

$$(0, 1) \setminus \{a\} = (0, a) \cup (a, 1)$$

is not connected. But 0 is not a cut point of $[0, 1]$ and so they are not homeomorphic. Note also that 1 is not a cut point.

(iv)

$$(0, 1) \quad [0, 1) \quad \text{and} \quad [0, 1].$$

We already saw that any interval is homeomorphic to one of these intervals. The only thing is to show that they are not homeomorphic, that is, we cannot discard any of these intervals. The idea is to just to count the number of points that are not cut points.

We already saw that every point of $(0, 1)$ is a cut point and $[0, 1]$ has two points, 0 and 1, that are not cut points. It is easy to see that 0 is not a cut point of $[0, 1)$. On the other hand, the same argument as in (iii) shows that every other point is a cut point. Thus all three intervals are not pairwise homeomorphic.

(v) First note that every point of $\mathbb{R}$ is a cut point.

On the other hand, no point of $\mathbb{R}^n$, $n > 1$ is a cut point. Let

$$X = \mathbb{R}^n \setminus \{p\}.$$

Pick two points q and r in $\mathbb{R}^n$. If p , q and r are not collinear then the line connecting q to r yields a path from q to r in X .

Now suppose that p , q and r are collinear. If $n > 1$ then we can find s , such that p , q and s are not collinear and p , r and s are not collinear. It follows that X is path-connected. But then X is connected.

Challenge Problems: (Just for fun)

6. If X is finite then it is obviously path-connected if it has at most one point. If it has two or more points then X is not connected and so it cannot be path-connected.

Now suppose the cardinality of X is at least the cardinality of the continuum (that is, the cardinality of the reals). If we pick two points a and b of X then we can find an injective function

$$f: [0, 1] \longrightarrow X. \quad \text{such that} \quad f(0) = a \quad \text{and} \quad f(1) = b.$$

The inverse image of a finite set is a finite set. This implies that the inverse image of a closed set is closed. But then f is continuous. Thus there is a path from a to b and X is path-connected.

Now suppose that X is countable. A continuous map

$$f: [0, 1] \longrightarrow X.$$

must have the property that the inverse image of every point is closed. Thus f corresponds to a decomposition of the unit interval into countably many closed sets.

7. (a) X is irreducible if and only if either X has less than two points or it has infinitely many points.

This is easy if X has zero or one point. If X has finitely many points and more than one, then X is not even connected, let alone irreducible. Now suppose that X is infinite. If $X = F \cup G$, where F and G are closed, then at least one of F and G is infinite. Let's suppose F is infinite. As F is closed, it follows that $F = X$. But then X is irreducible.

(b) Let $F = (-\infty, 0] \times \mathbb{R}^{n-1}$ and $G = [0, \infty) \times \mathbb{R}^{n-1}$. Then F and G are both closed, $\mathbb{R}^n = F \cup G$ but neither F nor G is the whole of $\mathbb{R}^n$.

(c) We already saw that I is an ideal.

Suppose that V is irreducible. Let f and g be two polynomials whose product lies in I , $fg \in I$. Then

$$V \subset V(f) \cup V(g).$$

Let $F = V \cap V(f)$ and let $G = V \cap V(g)$. Then

$$V = F \cup G$$

and F and G are closed. As V is irreducible it follows that $V = F$ or $V = G$. If $V = F$ then

$$V \subset V(f).$$

But then $f \in I$. Similarly the other way around.

Now suppose that I is prime. Suppose that

$$V = F \cup G,$$

where F and G are closed. Suppose that $V \neq G$. Then we may find $p \in V$ not in G .

Let $V \subset k^n$ be a Zariski closed subset, where k is a field. Let

$$I = I(V) = \{ f \in k[x_1, x_2, \dots, x_n] \mid f(p) = 0, \forall p \in V \}.$$

Show that I is an ideal in the polynomial ring $R = k[x_1, x_2, \dots, x_n]$.

Show that V is irreducible if and only if I is a prime ideal in R .

(d) There are a number of ways to see this. One can show directly that any linear polynomial $ax + by - c$ is irreducible. This is easy, as it has degree one. But a polynomial ring is a UFD and so the corresponding ideal

$$\langle ax + by - c \rangle$$

is prime.

We can also simply change coordinates. Translating, we may assume that $c = 0$. Then we may assume that $a = 0$ and $b = 1$. The key point is that these changes of coordinates are given by linear polynomials and so they induce homeomorphisms.

Now observe that the quotient ring

$$\frac{k[x, y]}{\langle y \rangle} \simeq k[x].$$

Indeed define a ring homeomorphism

$$\phi: k[x, y] \longrightarrow k[x]$$

by the rule

$$x \longrightarrow x \quad \text{and} \quad y \longrightarrow 0$$

It is easy to see that this map is surjective and that the kernel is $\langle y \rangle$. As $k[x]$ is an integral domain, it follows that $\langle y \rangle$ is a prime ideal.