

MODEL ANSWERS TO THE FIFTH HOMEWORK

1. (i) Pick an open subset $U \subset Y$. There are two cases. If $b \in U$ then

$$\begin{aligned} X &\supset f^{-1}(U) \\ &\supset f^{-1}(b) \\ &= X. \end{aligned}$$

Thus

$$f^{-1}(U) = X$$

which is open.

Otherwise, if $b \notin U$, then

$$f^{-1}(U) = \emptyset,$$

which is open.

Either way, the inverse image of U is open and f is continuous.

- (ii) It is enough to show that the inverse image of an open interval is open. We have

$$f^{-1}(a, b) = \left(\frac{a}{m}, \frac{b}{m}\right),$$

is open. Thus f is continuous.

- (iii) Probably the easiest way to proceed is to check the ϵ - δ condition (if you look at the inverse image of an interval you get the open strip between two lines of slope -1 . It is obvious this is open but to check this formally is not that clean).

We make things a little bit easier by working with open rectangles, rather than open balls.

Start with α and β . Pick $\epsilon > 0$. If we take

$$\delta = \frac{\epsilon}{2}$$

and

$$|\alpha' - \alpha| < \delta \quad \text{and} \quad |\beta' - \beta| < \delta$$

then

$$\begin{aligned}
 |a(\alpha', \beta') - a(\alpha, \beta)| &= |\alpha' + \beta' - \alpha - \beta| \\
 &\leq |\alpha' - \alpha| + |\beta' - \beta| \\
 &< \delta + \delta \\
 &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon.
 \end{aligned}$$

Thus a is continuous.

(iv) Let

$$h: X \longrightarrow \mathbb{R}^2$$

be the function

$$h(x) = (f(x), g(x)).$$

Then $h(x)$ is continuous as $f = p \circ h$ and $g = q \circ h$ are continuous. The composition of h and a is the function $f + g$, and so $f + g$ is continuous.

(v) Probably the easiest way to proceed is to check the ϵ - δ condition (if you look at the inverse image of an interval you get the disjoint union of two open strips between two hyperbolae).

We make things a little bit easier by working with open rectangles, rather than open balls.

Suppose that we start with α and β . Pick $\epsilon > 0$. There is no harm in assuming that $\epsilon < 1$. Let

$$\gamma = \max(|\alpha|, |\beta|) + 1.$$

If we take

$$\delta_1 = \frac{\epsilon}{2\gamma} \quad \text{and} \quad \delta_2 = \frac{\epsilon}{2\gamma}$$

and

$$|\alpha' - \alpha| < \delta_1 \quad \text{and} \quad |\beta' - \beta| < \delta_2$$

then

$$\max(|\alpha|, |\beta|, |\alpha'|, |\beta'|) \leq \gamma$$

and

$$\begin{aligned}
|m(\alpha', \beta') - m(\alpha, \beta)| &= |\alpha'\beta' - \alpha\beta| \\
&= |\alpha'\beta' - \alpha\beta' + \alpha\beta' - \alpha\beta| \\
&\leq |\alpha'\beta' - \alpha\beta'| + |\alpha\beta' - \alpha\beta| \\
&\leq |\beta'| |\alpha' - \alpha| + |\alpha| |\beta' - \beta| \\
&< \gamma(\delta_1 + \alpha\delta_2) \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus m is continuous.

(vi) This is very similar to (iv). The function fg is the composition of h and m and so fg is continuous.

(vii) We proceed by induction on n . If $n = 0$ we have the constant map, $x \rightarrow 1$ and we saw that was continuous in (i). If $n = 1$ we have the identity map and this is continuous. If $f = x \rightarrow x^n$ is continuous then $x \rightarrow x^{n+1}$ is the function xf which is continuous by (vi).

(viii) The function $x \rightarrow rx^n$ is continuous by (i), (vi) and (viii) (or indeed by (ii) and (viii)). Hence the polynomial map is continuous by induction and (iv).

(ix) It suffices to show that the functions

$$f_+ : (0, \infty) \rightarrow \mathbb{R} \quad \text{given by} \quad f_+(x) = \frac{1}{x}$$

and

$$f_- : (-\infty, 0) \rightarrow \mathbb{R} \quad \text{given by} \quad f_-(x) = \frac{1}{x}$$

are continuous. Indeed the intersection of the two open sets $(0, \infty)$ and $(-\infty, 0)$ is empty and so we can just patch together f_+ and f_- .

Note that the image of f_+ lands in $(0, \infty)$. To show that f_+ is continuous, it is enough to show that the inverse image of an open interval inside $(0, \infty)$ is open. If we start with $(a, b) \subset (0, \infty)$ then

$$f^{-1}(a, b) = \left(\frac{1}{b}, \frac{1}{a}\right),$$

which is an open interval (or the open ray $(1/b, \infty)$ if $a = 0$). Thus f_+ is continuous. The argument for f_- is similar (or we could use the fact that f_- is the composition of the map $x \rightarrow -x$, which is continuous by (ii) and f_+). Thus f is continuous.

(x) As the set $\{0\}$ is closed and g is continuous, the inverse image of 0 is closed. But then U , the complement, is open.

Now we proceed as in (iv) and (vi). The function ϕ is the composition of the natural diagonal map

$$U \longrightarrow U \times U \quad \text{given by} \quad x \longrightarrow (x, x)$$

which is continuous, the natural inclusion

$$U \times U \longrightarrow \mathbb{R}^n \times \mathbb{R}^n.$$

which is continuous, the map

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^2$$

given by

$$(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) \longrightarrow (f(x_1, x_2, \dots, x_n), g(y_1, y_2, \dots, y_n))$$

which is continuous, as the components are continuous by (viii), the function

$$\mathbb{R} \times \mathbb{R}^* \longrightarrow \mathbb{R} \times \mathbb{R} \quad \text{given by} \quad (x, y) \longrightarrow (x, \frac{1}{y})$$

(which is continuous as its components are continuous by (ix)) and the multiplication map m , which is continuous. By our choice of U , the image of the composition of the first three maps lands in $\mathbb{R} \times \mathbb{R}^*$. Thus ϕ is continuous, as it is the composition of continuous functions.

2. If g is continuous then f is continuous, as it is the composition of g and the natural inclusion j , which is continuous.

Now suppose that f is continuous. If W is an open subset of B then we may find V open in Y such that $W = B \cap V$. We have

$$\begin{aligned} g^{-1}(B \cap V) &= f^{-1}(V) \\ &= U. \end{aligned}$$

U is open as f is continuous. It follows that g is continuous.

3. (i) False. Let $X = \{a, b\}$ be a set with two elements. Then the identity $i_X: X \longrightarrow X$ is continuous, if we give the domain the discrete topology and the image the trivial topology, since the discrete topology is finer than the trivial topology. But these two spaces are not homeomorphic and so i_X is not a homeomorphism.

(ii) True. Let $B \subset Y$ be a closed set. Then $V = Y \setminus B$ is open. As f is continuous it follows that $U = f^{-1}(V)$ is open. As $f(U) \subset V$ it follows that $A = f^{-1}(B)$ is the complement of U . But then A is closed.

(iii) True. This is just (ii), in reverse. Let $V \subset Y$ be an open set. Then $B = Y \setminus V$ is closed. By assumption $A = f^{-1}(B)$ is closed. As $f(A) \subset B$ it follows that $U = f^{-1}(V)$ is the complement of A . But then U is open and so f is continuous.

(iv) False (of course). Take the interval $(-1/2, 1/2)$. The inverse image of this is $(-\infty, 0]$, which is not open.

(v) False. Let $i: (0, 1) \rightarrow \mathbb{R}$ be the natural inclusion. The i is a continuous map. $(0, 1)$ is a closed subset of $(0, 1)$. But its image is not closed, $(0, 1) \subset \mathbb{R}$ is not closed.

(vi) True. Recall that a point $b \in X$ belongs to the closure of A if and only if every neighbourhood of b intersects A .

Let b be a point of the closure of A and let $c = f(b)$. Let $V \subset Y$ be a neighbourhood of c . By assumption the interior of V contains c . The inverse image of the interior of V is open, as f is continuous, and so $U = f^{-1}(V)$ is a neighbourhood of a .

As a belongs to the closure of A it follows that U intersects A . But then V intersects $f(A)$ and so c belongs to the closure of $f(A)$.

4. (i) We saw in (viii) that any polynomial is continuous. In particular the function

$$\mathbb{R}^n \rightarrow \mathbb{R} \quad \text{given by} \quad \|x\|^2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 + x_{n+1}^2,$$

is continuous. The singleton set $\{1\} = [1, 1]$ is closed and the sphere is the inverse image of this set.

N is a point of the sphere as

$$0^2 + 0^2 + \cdots + 0^2 + 1^2 = 1.$$

The set $\{N\}$ is a closed subset of the sphere as it is a closed subset of $\mathbb{R}^{n+1}$. Thus U is an open subset. Define a map

$$a: \mathbb{R}^n \rightarrow \mathbb{R}^{n+1} \quad \text{given by} \quad (x_1, x_2, \dots, x_n) \rightarrow (x_1, x_2, \dots, x_n, 0)$$

a is continuous as its components are continuous functions. Now consider the map

$$b: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \quad \text{given by} \quad (x_1, x_2, \dots, x_{n+1}) \rightarrow (x_1, x_2, \dots, x_n).$$

b is also continuous as its components are continuous functions. The restriction $b|_H$ is continuous and it is the inverse of a . Thus H is homeomorphic to $\mathbb{R}^n$.

(ii) The line connecting N to $P = (x_1, x_2, \dots, x_{n+1})$ is given parametrically as

$$tP + (1 - t)N = (tx_1, tx_2, tx_3, \dots, tx_{n-1}, tx_n, (1 - t) + tx_{n+1})$$

We want the value of t so that we get a point of the hyperplane H :

$$(1 - t) + tx_{n+1} = 0 \quad \text{so that} \quad t = \frac{1}{1 - x_{n+1}}.$$

The point Q is then

$$\left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \frac{x_3}{1-x_{n+1}}, \dots, \frac{x_{n-1}}{1-x_{n+1}}, \frac{x_n}{1-x_{n+1}}, 0\right).$$

The only point where $x_{n+1} = 1$ is at N . Since we excluded this point, f is continuous, as its components are rational functions.

(iii) We have to write down the inverse function

$$g: H \longrightarrow U.$$

Given a point $Q \in H$ consider the line connecting Q to N . This line intersects the sphere in one point P (the defining equation of the sphere is quadratic and when we restrict this to a line we get a quadratic polynomial in one variable. A quadratic polynomial has two roots. One is given by N and the other by P).

The line connecting $(x_1, x_2, \dots, x_n, 0) \in H$ to N is given parametrically by

$$tQ + (1-t)N = (tx_1, tx_2, tx_3, \dots, tx_{n-1}, tx_n, (1-t)).$$

For this to lie on the sphere we must have

$$\begin{aligned} 1 &= (tx_1)^2 + (tx_2)^2 + \dots + (tx_n)^2 + (1-t)^2 \\ &= t^2x_1^2 + t^2x_2^2 + \dots + t^2x_n^2 + 1 - 2t + t^2 \\ &= t^2(x_1^2 + x_2^2 + \dots + x_n^2 + 1) + 1 - 2t. \end{aligned}$$

It follows that

$$t^2(x_1^2 + x_2^2 + \dots + x_n^2 + 1) - 2t = 0.$$

The solution $t = 0$ corresponds to N . Otherwise

$$t = \frac{2}{x_1^2 + x_2^2 + \dots + x_n^2 + 1}$$

(as predicted we get a quadratic equation for t . One root corresponds to N and the other root corresponds to P). Thus

$$g(x_1, x_2, \dots, x_n, 0) = (tx_1, tx_2, tx_3, \dots, tx_{n-1}, tx_n, (1-t)).$$

It is clear from the geometric description of f and g that g is the inverse of f . g is continuous as its components are rational functions and the denominator

$$x_1^2 + x_2^2 + \dots + x_n^2 + 1$$

is never equal to zero.

Thus f is a homeomorphism.

5. At first sight this question looks very hard. Let's start with a seemingly much easier problem. In fact after we solve this problem we are basically done.

Consider using polar coordinates. The closed unit ball, a closed disk, is given in polar coordinates (r, θ) by the two conditions

$$r \in [0, 1] \quad \text{and} \quad \theta \in [0, 2\pi).$$

Here is the easier problem. First of all forget about $U_{a,b}$ and $C_{a,b}$. Note that a and b are just the angles in radians. Imagine the sector of a disk $D_{a,b}$, given by the two conditions

$$r \in [0, 1] \quad \text{and} \quad \theta \in [a, b].$$

(If you draw a picture, you just get a slice of cake or a slice of pizza. Depending on a and b , it might look more like you took a slice of cake or pizza).

At this point we can even drop the condition on r . We just get the region bounded by two half lines through the origin,

$$\theta \in [a, b].$$

Call this region $R_{a,b}$. To show that all of these are homeomorphic, it suffices to show that $R_{a,b}$ is homeomorphic to $R_{0,1}$.

We can construct a continuous map using a linear function on θ ,

$$f: R_{0,1} \longrightarrow R_{a,b}$$

using the rule

$$f(r, \theta) = (r, (b - a)\theta + a).$$

There is one issue to deal with. We have clearly written down a continuous function of r and θ . We have to check this translates to a continuous function of x and y . This is not hard, using the usual way to get from polar coordinates to cartesian coordinates.

Note that we have an inverse function

$$g: R_{a,b} \longrightarrow R_{0,1} \quad \text{given by} \quad g(r, \theta) = (r, \frac{\theta - a}{b - a}).,$$

which is clearly continuous. Thus f is a homeomorphism.

Now note that we can patch together two such functions. One that identifies $R_{0,1}$ with $R_{a,b}$ and the other that identifies $R_{1,2\pi=0}$ with $R_{b,a}$. These two homeomorphisms agree on the overlap, so that we get a continuous bijection

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

The inverse function has the same shape and so this is a homeomorphism.

This restricts to the required homeomorphism between $U_{0,1}$ and $U_{a,b}$. It follows that $U_{a,b}$ and $U_{c,d}$ are homeomorphic (at least if $(a - b)(c - d) \neq 0$).

Challenge Problems: (Just for fun)

6. Let V be a Zariski closed subset. Then V is the zero locus of finitely many polynomials $f_1, f_2, \dots, f_k$. In this case V is the intersection of hypersurfaces, Zariski closed subsets which are the zero locus of a single polynomial equation. Suppose that the polynomial is f . f determines a function

$$\phi: k^n \longrightarrow k \quad \text{given by} \quad x \longrightarrow f(x).$$

This function is continuous in the Euclidean topology and V is the inverse image of the closed subset $\{0\} = [0, 0]$. As the inverse image of a closed subset is closed, it follows that V is closed in the Euclidean topology.

But then the Zariski topology is coarser than the Euclidean topology.

7. Let $Y \subset \mathbb{R}$ be a subset which is homeomorphic to an ordinal α , with the order topology,

$$f: \alpha \longrightarrow Y.$$

Suppose that $\beta < \alpha$. Let $b = f(\beta)$ and $c = f(\beta + 1)$. Then $U_\beta = (b, c) \subset \mathbb{R} \setminus Y$ is an open interval. The set

$$\{U_\beta \mid \beta < \alpha\}$$

of all open intervals we get this way is pairwise disjoint. As in 6 (ii) the number of such discs is countable (or finite). Thus α is countable (or finite).