

MODEL ANSWERS TO THE FOURTH HOMEWORK

1. We are going to have to be systematic about this, as with the first homework.

Let x be the number of open subsets with one element, y the number with two and z the number with three.

Note that y ranges between 0 and 6. This is the most useful invariant to keep track of as it takes on the most values and there are quite examples to deal with.

Case: $y = 0$.

In this case $0 \leq x \leq 1$ and $0 \leq z \leq 1$, since the union of two singleton sets has two elements and the intersection of two sets with three elements is a set with two elements. If $x = z = 0$ then we have the trivial topology. If $x = 1$ and $z = 0$ then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a\}, X \}$$

If $x = 0$ and $z = 1$ then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a, b, c\}, X \}$$

If $x = 1$ and $z = 1$ then up to homeomorphism there are two cases: the singleton subset is a subset of the set with three elements,

$$\mathcal{T} = \{ \emptyset, \{a\}, \{a, b, c\}, X \}$$

or it does not

$$\mathcal{T} = \{ \emptyset, \{a\}, \{b, c, d\}, X \}.$$

Case: $y = 1$.

In this case $0 \leq x \leq 2$ and $0 \leq z \leq 2$, since the pairwise union of three sets with one element gives three sets with two elements and the pairwise intersection of three sets with three elements gives three sets with two elements.

If $x = z = 0$ then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a, b\}, X \}$$

If $x = 1$ and $z = 0$ then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a\}, \{a, b\}, X \}$$

since otherwise the union of the singleton set and the two element set is a set with three elements. Similarly if $x = 0$ and $z = 1$ then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a, b\}, \{a, b, c\}, X \}.$$

Now suppose that $x = 2$ and $z = 0$. Then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, X \}.$$

Similarly if $x = 0$ and $z = 2$ then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X \}.$$

Now suppose that $x = 1$ and $z = 1$. Then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X \}.$$

Now suppose that $x = 2$ and $z = 1$. Then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X \}.$$

Similarly if $x = 1$ and $z = 2$ then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X \}.$$

Finally suppose that $x = z = 2$. Then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X \}.$$

Case: $y = 2$.

In this case $0 \leq x \leq 2$ and $0 \leq z \leq 2$. If $x = z = 0$ then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a, b\}, \{c, d\}, X \}.$$

Note that we cannot have $x = 1$ and $z = 0$ or vice-versa. If $x = z = 1$ then up to homeomorphism we have two cases. The two open sets with two elements intersect

$$\mathcal{T} = \{ \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X \}.$$

or they are disjoint

$$\mathcal{T} = \{ \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X \}.$$

Now suppose that $x = 2$ and $z = 1$. Then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X \}.$$

Similarly if $x = 1$ and $z = 2$ then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, X \}.$$

If $x = 2$ and $z = 2$ then up to homeomorphism we have

$$\mathcal{T} = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X \}.$$

Case: $y = 3$.

In this case $1 \leq x \leq 3$ and $1 \leq z \leq 3$. Note that up to a bijection, there are three ways that three sets of two elements can be distributed:

$$\begin{aligned} & \{a, b\}, \{c, d\}, \{a, c\} \\ & \{a, b\}, \{a, c\}, \{b, c\} \\ & \{a, b\}, \{a, c\}, \{a, d\}. \end{aligned}$$

In the first way two sets are disjoint, in the second each pair of sets intersects and there are no common elements and in the third the three sets contain a common element.

If we have a topology that contains the first collection then it must contain $\{a\}$, $\{c\}$, $\{a, b, c\}$ and $\{a, c, d\}$. It is then easy to see it cannot contain more singleton or three element sets. Up to homeomorphism we then get

$$\mathcal{T} = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}.$$

If we have a topology that contains the second collection then it must contain $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b, c\}$. It is then easy to see it cannot contain more singleton or three element sets. Up to homeomorphism we then get

$$\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\}.$$

If we have a topology that contains the third collection then it must contain $\{a\}$, $\{a\}$, $\{a, b, c\}$, $\{a, b, d\}$ and $\{a, c, d\}$. It is then easy to see it cannot contain more singleton or three element sets. Up to homeomorphism we then get

$$\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}.$$

Case: $y = 4$. Then, up to bijection, we are missing $\{a, b\}$ and $\{c, d\}$ or $\{a, b\}$ and $\{a, c\}$ (that is, the two sets with two elements we are missing are disjoint or share one element). In the first case we must contain

$$\{a\} = \{a, c\} \cap \{a, d\}.$$

and by symmetry we must contain every other singleton subset. Thus we have the discrete topology.

In the second case we must contain

$$\{b\} = \{b, c\} \cap \{b, d\}.$$

and by symmetry we must contain

$$\{b\}, \quad \{c\} \quad \text{and} \quad \{d\}.$$

It then follows that we can get every set with three elements that contains d . Up to homeomorphism we then get

$$\mathcal{T} = \{ \emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X \}.$$

Case: $y = 5$. This is not possible. As in case (4) we must have the discrete topology.

Case: $y = 6$. In this case we must have the discrete topology.

2. X and Y are homeomorphic if and only if they have the same cardinality. Indeed if X and Y are homeomorphic then they certainly have the same cardinality. Now suppose that X and Y have the same cardinality. Let $f: X \rightarrow Y$ be a bijection. Then f takes finite sets to finite sets and so does the inverse of f . It follows that f induces a bijection between the closed sets. But then f induces a bijection between the open sets and f is a homeomorphism.

3. It suffices to show that any closed and bounded interval $[a, b]$ is homeomorphic to $[0, 1]$. We proceed as in the proof of Theorem 7.2. Let

$$f: [0, 1] \rightarrow [a, b]$$

be given by the rule

$$f(x) = a + (b - a)x.$$

A basis for the subspace topology on $[0, 1]$ is given by open intervals, intervals of the form $[0, \beta)$, intervals of the form $(\alpha, 1]$ and $[0, 1]$. A basis for the subspace topology on $[a, b]$ is given by open intervals, intervals of the form $[a, \beta)$, intervals of the form $(\alpha, b]$ and $[a, b]$.

We already saw that an open interval is sent to an open interval. The interval $(\alpha, 1]$ is sent to $(a + \alpha(b - a), b)$, which is open in $[a, b]$. Similarly for the other two types of intervals.

The inverse of f is the function

$$g: (a, b) \rightarrow (0, 1) \quad \text{given by} \quad g(x) = \frac{x - a}{b - a}.$$

We already saw that g takes open intervals to open intervals. g takes $[a, \beta)$ to the open interval

$$[0, \frac{\beta - a}{b - a}).$$

Similarly for the two other types of intervals. Thus f is a homeomorphism, since it induces a bijection between two bases.

4. We have to show that $(0, \infty)$ is homeomorphic to (a, ∞) to $(-\infty, a)$ and to $(-\infty, \infty)$. Consider the map

$$f: (0, \infty) \rightarrow (a, \infty) \quad \text{given by} \quad x \rightarrow x + a.$$

This sends the open interval (α, β) to $(a + \alpha, a + \beta)$. The inverse is the map

$$g: (a, \infty) \longrightarrow (0, \infty) \quad \text{given by} \quad x \longrightarrow x - a$$

This sends the open interval (α, β) to $(-a + \alpha, -a + \beta)$. Thus f is a bijection that induces a correspondence between two bases. Thus f is a homeomorphism.

Now consider the map

$$f: (a, \infty) \longrightarrow (-\infty, -a) \quad \text{given by} \quad x \longrightarrow -x.$$

This sends (α, β) to $(-\beta, -\alpha)$. The inverse of f is the map

$$g: (-\infty, -a) \longrightarrow (a, \infty) \quad \text{given by} \quad x \longrightarrow -x.$$

Thus f is a homeomorphism.

It remains to show that $(0, \infty)$ and $(-\infty, \infty)$ are homeomorphic. It suffices to show that $(-1, 1)$ and $\mathbb{R}$ are homeomorphic. We patch together two homeomorphisms, one from $(0, 1)$ to $(0, \infty)$ and the other from $(-1, 0)$ to $(-\infty, 0)$.

Consider the function

$$f: (-1, 1) \longrightarrow (-\infty, \infty) \quad \text{given by} \quad x \longrightarrow \frac{x}{1 - |x|}.$$

Then f sends the interval (a, b) to the interval

$$\left(\frac{a}{1 - |a|}, \frac{b}{1 - |b|}\right)$$

The inverse map is given by

$$g: (-\infty, \infty) \longrightarrow (-1, 1) \quad \text{given by} \quad x \longrightarrow \frac{x}{1 + |x|}.$$

This sends the interval (a, b) to

$$\left(\frac{a}{1 + |a|}, \frac{b}{1 + |b|}\right).$$

Thus f is a homeomorphism.

5. True or false? If true then give a proof and if false then give a counterexample.

(i) True. Define a function

$$f: X \times Y \longrightarrow Y \times X \quad \text{by the rule} \quad (x, y) \longrightarrow (y, x).$$

Note if U is open in X and V is open in Y then f takes the open set $U \times V$ to the open set $V \times U$. The inverse is the map

$$g: Y \times X \longrightarrow X \times Y \quad \text{given by} \quad (y, x) \longrightarrow (x, y).$$

By symmetry g takes a basis element to a basis element. Thus f is a homeomorphism.

(ii) False. These topological spaces are not homeomorphic. Given any two points of $\mathbb{R}$ with the Euclidean topology, we may find an open set containing one but not the other (in fact much more is true, the Euclidean topology is Hausdorff).

This is not true of the topology with open sets (a, ∞) . If $0 \in (a, \infty)$ then $a < 0$. But then $1 \in (a, \infty)$. Thus there is no open set containing 0, not containing 1.

(iii) True. Define a function

$$f: \mathbb{R} \longrightarrow \mathbb{R} \quad \text{by the rule} \quad x \longrightarrow -x.$$

This is clearly a bijection. In fact f is its own inverse. f takes the open set (a, ∞) to the open set $(-\infty, -a)$. Thus f induces a bijection on open sets and so f is a homeomorphism.

(iv) True. The Hausdorff property may be stated only in terms of open sets and so it must be preserved by homeomorphism. It is also easy to check this directly.

(v) False. The trick is to use a topology that only sees the cardinality. The two obvious choices are the discrete topology or the trivial topology.

In this case $X \times Y$ has the discrete topology (or the trivial topology). Similarly for $Z \times W$.

Now take $X = Y$ a set with two elements, Z a set with four elements and W as set with one element. Then $X \times Y$ and $Z \times W$ have four elements and so they are homeomorphic.

But X is not homeomorphic to either Z or W , since they don't have the same cardinality.

In fact for an more extreme example, we could even take $Y = W$ the empty set. In this case $X \times Y$ and $Z \times W$ are the empty set, and so homeomorphic, regardless of X and Z .

Challenge Problems: (Just for fun)

6. (i) Easy. Concentric circle centred at the origin are disjoint and uncountable, since the radius can be any positive real.

(ii) Suppose that U_α is a collection of disjoint disks. Given a disc U_α , we may find a point

$$(x_\alpha, y_\alpha) \in U_\alpha$$

with rational coordinates. Let Λ be the indexing set for the alphas. We have constructed a function

$$\phi: \Lambda \longrightarrow \mathbb{Q}^2.$$

This sends the label α to the point

$$\phi(\alpha) = (x_\alpha, y_\alpha).$$

By assumption, if $\alpha \neq \beta$ then U_α and U_β are disjoint. In particular

$$(x_\alpha, y_\alpha) \neq (x_\beta, y_\beta).$$

But then

$$\phi(\alpha) \neq \phi(\beta)$$

so that ϕ is injective.

But

$$\mathbb{Q}^2 \subset \mathbb{R}^2$$

is countable. Thus Λ is countable (or finite, even), since it injects into a countable set.

For the purists, notice that it looks as though we invoke the axiom of choice when we choose (x_α, y_α) (this certainly involves a lot of choices, as Λ might be quite large). This can be avoided as follows.

Given U_α consider the smallest positive integer $r > 0$ such that we can find a point $(x, y) \in U_\alpha$ such that (rx, ry) has integer coordinates (in other words, we take the denominator of the coordinates x and y to be as small as possible), the smallest r such that

$$U_\alpha \cap \frac{1}{r}\mathbb{Z}^2$$

is non-empty. This does not involve any choices. Once r has been chosen, the possible values X_r of x are discrete, they belong to the set

$$X_r \subset \left\{ \frac{m}{r} \mid m \in \mathbb{Z} \right\}.$$

The possible x -coordinates X_r are bounded from below, as well. We take the smallest possible value for x , the smallest element of X_r . Having chosen x , we repeat the same procedure for y . In the end, we are just picking the bottom, left most point of U_α (since U_α is a disk, we could even have picked x and y independently). The resulting point

$$(x_\alpha, y_\alpha)$$

has rational coordinates and does not involve any choices. So we don't need to invoke the axiom of choice.