

MODEL ANSWERS TO THE THIRD HOMEWORK

1. Let B be the set of accumulation points that don't belong to A . We want to show

$$A \cup B = \overline{A}$$

We first show the LHS is contained in the RHS. Suppose that $b \in A \cup B$. If $b \in A$ then $b \in \overline{A}$ and b belongs to the RHS. If $b \in B$ and $b \notin A$ then b belongs to the closure of

$$A \setminus \{b\} = A.$$

Now let $b \in \overline{A}$. If $b \in A$ then b certainly belongs to the LHS. Now suppose that $b \notin A$. As

$$A \setminus \{b\} = A$$

it follows that b is an accumulation point of A , so that $b \in B$. Thus b again belongs to the LHS. Thus the RHS belongs to the LHS and we have equality.

2. We have to show that every open set in the product topology is a union of basis elements. In fact it suffices to show that if U is open in X and V is open in Y then $U \times V$ is a union of elements of $\mathcal{D}$, since subsets of this form are a basis for the product topology.

By assumption we may find $U_i \in \mathcal{B}$ and $V_j \in \mathcal{C}$ such that

$$U = \bigcup_i U_i \quad \text{and} \quad V = \bigcup_j V_j.$$

Note that

$$U_i \times V_j \in \mathcal{D}.$$

We have

$$\begin{aligned} \bigcup_{i,j} U_i \times V_j &= \bigcup_i \left(\bigcup_j U_i \times V_j \right) \\ &= \bigcup_i U_i \times V \\ &= U \times V. \end{aligned}$$

Thus $U \times V$ is the union of elements of $\mathcal{D}$.

3. Suppose that (i) holds. Let F and G be two disjoint neighbourhood of x and y . Let U be the interior of F and let V be the interior of G . Then $x \in U$ and $y \in V$. U and V are certainly disjoint, as F and G are disjoint. Thus (ii) holds.

Now suppose that (ii) holds. Let U and V be two disjoint open subsets such that $x \in U$ and $y \in V$. Let A be the complement of V . As V does not intersect U , it follows that A contains U . A is closed as V is open. $y \in V$ and so $y \notin A$.

$$x \in U \subset A$$

and so A is a closed neighbourhood of x , not containing y . Thus (iii) holds.

Now suppose that (iii) holds. Suppose that A is a closed neighbourhood of x not containing y . Let $F = A$ and let G be the complement of A . F and G are disjoint, as they are complements. F is a neighbourhood of x . G is open as A is closed and $y \in G$. It follows that G is a neighbourhood of y . Thus (i) holds.

It follows that (i), (ii) and (iii) are equivalent.

To finish, we will show that (ii) and (iv) are equivalent. Suppose that (ii) holds. Let W be the complement of the diagonal Δ . Suppose that $p \in W$. Then $p = (x, y)$, $x \neq y \in X$. As we are assuming (ii), we may find U and V open and disjoint, so that $x \in U$ and $y \in V$. Then

$$p = (x, y) \in U \times V \subset W$$

The last inclusion follows as U and V are disjoint. As U and V are open, $U \times V$ is open in $X \times Y$. But then W is open, as p is an arbitrary point of W . It follows that Δ is closed.

Now suppose that Δ is closed. Then the complement W is open. Suppose that $x \neq y \in X$. Then $p = (x, y) \in W$. As W is open, it is union of basis elements. Thus we may find U and V open so that

$$p = (x, y) \in U \times V \subset W.$$

It follows that $x \in U$, $y \in V$ and U and V are disjoint, by the last inclusion. Thus (ii).

4. (i) False. Let $X = \mathbb{R}$ and $Y = [0, 1]$. Then $U = Y$ is open in Y but not in X .

(ii) True. Let $x \in X$ and let A be the closure of $\{x\}$. Then $x \in A$ and A is the smallest closed subset of X containing x . Suppose that $y \neq x \in X$. By 3, we may find a closed neighbourhood B of x not containing y . As $x \in B$ and B is closed, $A \subset B$. But then $y \notin A$. Thus $A = \{x\}$ is closed.

(iii) False. Let X be any infinite set, and let $\mathcal{T}$ be the topology where the closed sets are the finite sets, plus X . Then every singleton set is closed, since it is finite. Let $x \neq y$. Suppose that U and V are two open sets. Then their complements are finite. The union is finite and so $U \cap V$ is non-empty. It follows that there every pair of non-empty

open sets must intersect. But then we cannot separate x and y using open sets.

(iv) True. Suppose that $p \neq q \in X \times Y$. Then $p = (a, b)$ and $q = (c, d)$. Since $p \neq q$, either $a \neq c$ or $b \neq d$. Suppose that $a \neq c$. As X is Hausdorff, we may find disjoint open sets U and V open in X such that $a \in U$ and $c \in V$. Then

$$p = (a, b) \in U \times Y \quad \text{and} \quad q = (c, d) \in V \times Y.$$

Note that $U \times Y$ and $V \times Y$ are open and disjoint. There is a similar argument if $b \neq d$. It follows that $X \times Y$ is Hausdorff.

(v) True. We first check that if F and G are closed then $F \times G$ is closed. Let U be the complement of F . Then the complement of $F \times Y$ is $U \times Y$, which is open. But then $F \times Y$ is closed. Similarly $X \times G$ is closed. But then

$$F \times G = (F \times Y) \cap (X \times G)$$

is closed.

Now

$$A \subset \overline{A} \quad \text{and} \quad B \subset \overline{B}$$

so that

$$A \times B \subset \overline{A} \times \overline{B}.$$

As $\overline{A} \times \overline{B}$ is closed, it follows that the LHS is contained in the RHS,

$$\overline{A \times B} \subset \overline{A} \times \overline{B}$$

To prove the reverse direction, recall the following characterisation of the closure: a point p belongs to the closure of a set C if every neighbourhood of the point p intersects C .

Suppose that

$$(a, b) \in \overline{A} \times \overline{B}.$$

Let W be a neighbourhood of (a, b) . Then

$$(a, b) \in U \times V$$

for two open sets U and V , as sets of the form $U \times V$ are a basis for the product topology. Since $a \in \overline{A}$, U and A must intersect. Similarly, V and B must intersect. Hence W must intersect $A \times B$. Thus (a, b) belongs to the LHS and we have equality.

(vi) True. Let X be a Hausdorff topological space and let $Y \subset X$ be any subset. Suppose that $a \neq b \in Y$. Then $a \neq b \in X$ and so we may find two disjoint open subsets U and V of X such that $a \in U$ and $b \in V$. Then $U_1 = U \cap Y$ and $V_1 = V \cap Y$ are two disjoint open subsets of Y such that $a \in U_1$ and $b \in V_1$. It follows that Y is Hausdorff.

5. Let $x \neq y \in X$. If $r = d(x, y)$ then $r > 0$. Let

$$U = B_{r/2}(x) \quad \text{and} \quad V = B_{r/2}(y).$$

It is clear that $x \in U$ and $y \in V$ are both open. Suppose that $z \in U \cap V$. Then

$$\begin{aligned} r &= d(x, y) \\ &\leq d(x, z) + d(z, y) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r. \end{aligned}$$

As this is a contradiction, it must be impossible to find $z \in U \cap V$. But then U and V are disjoint and X is Hausdorff.

Challenge Problems: (Just for fun)

6. (i) We already saw that the closed sets of k are the finite sets, plus k . If k is infinite (for example, the reals or the complexes) then we saw in 4 (iii) is not Hausdorff.

(ii) Consider what are the closed sets in the product topology (say, in $\mathbb{R}^2$). If we take a finite set in the first factor and take the product with second factor then we get finitely many vertical lines. Similarly if we take a finite set in the second factor and take the product with the first factor then we get finitely many horizontal lines. If we take the union of two such sets then we get finitely many horizontal and vertical lines. If we intersect arbitrarily many of these we can only get finitely many horizontal and vertical lines and finitely many points.

By definition of the product topology, it follows that the only closed sets in the product topology are the union of finitely many points, finitely many horizontal lines and finitely many vertical lines.

But there are many more subvarieties. For example, line

$$y = x$$

or indeed any line that is neither horizontal nor vertical, or the hyperbola

$$xy = 1$$

or the circle

$$x^2 + y^2 = 1.$$

The list goes on.

(iii) Let the variables in the first factor be $x_1, x_2, \dots, x_n$ and let the variables in the second factor be $y_1, y_2, \dots, y_n$. We want to write down equations such that if

$$p = (a_1, a_2, \dots, a_n) \quad \text{and} \quad q = (b_1, b_2, \dots, b_n)$$

then $p = q$, so that

$$(p, q) = (a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n) = (p, p) = (a_1, a_2, \dots, a_n, a_1, a_2, \dots, a_n)$$

Put this way, the equations we want are clear

$$x_1 - y_1 = 0 \quad x_2 - y_2 = 0 \quad x_3 - y_3 = 0 \quad \dots \quad x_n - y_n = 0.$$

(iv) Note that the image of k^m is closed. Indeed it is given by the equations

$$x_{n-m+1} = 0 \quad x_{n-m+2} = 0 \quad x_{n-m+3} = 0 \quad \dots \quad x_n = 0.$$

So $F \subset k^m$ is closed in the subspace topology if and only if its image in k^n is closed in the Zariski topology.

Now if $g(x_1, x_2, \dots, x_m)$ is a polynomial in the first m variables it defines a polynomial in the first n variables, in an obvious way. Vice-versa, if $f(x_1, x_2, \dots, x_n)$ is a polynomial in the variables $x_1, x_2, \dots, x_n$ then let

$$g(x_1, x_2, \dots, x_m) = f(x_1, x_2, \dots, x_m, 0, 0, \dots, 0),$$

be the polynomial in the variables $x_1, x_2, \dots, x_m$ you get by setting the last $n - m$ coordinates equal to zero. Note that

$$g(a_1, a_2, \dots, a_m) = 0 \quad \text{if and only if} \quad f(a_1, a_2, \dots, a_m, 0, 0, \dots, 0) = 0.$$

Thus $V \subset k^m$ is the zero locus of polynomials g_i in the first m variables if and only if $V \subset k^n$ is the zero locus of polynomials f_i in the first n variables.

Thus the subspace topology and the Zariski on k^m coincide.

(iv) First observe that k^n is never Hausdorff. Indeed, it contains the subspace $k = k^1$ and this is not Hausdorff by (i). By 4 (vi) it follows that k^n is not Hausdorff.

We saw in (3) that this means that the diagonal in k^{2n} is not closed, in the product topology. But the diagonal is closed in the Zariski topology. So the two topologies are different.