

MODEL ANSWERS TO THE SECOND HOMEWORK

1. (i) The open subsets are complements of finite sets plus the empty set. There are two cases. If Y is the complement of a finite set then Y is open. Otherwise the only open set contained in Y is the empty set.

$$\text{int}(Y) = \begin{cases} Y & \text{if the complement of } Y \text{ is finite,} \\ \emptyset & \text{otherwise.} \end{cases}$$

(ii) The closed subsets are the finite subsets, plus X . If Y is finite then it is closed. If Y is infinite then the only closed set containing Y is X

$$\overline{Y} = \begin{cases} Y & \text{if } Y \text{ is finite,} \\ X & \text{otherwise.} \end{cases}$$

(iii) Using (i) and (ii) there are four cases.

$$\partial Y = \begin{cases} \emptyset & \text{if } X \text{ is finite,} \\ Y & \text{if } Y \text{ is finite and the complement of } Y \text{ is infinite,} \\ X \setminus Y & \text{if the complement of } Y \text{ is finite and } Y \text{ is infinite,} \\ X & \text{both } Y \text{ and the complement of } Y \text{ are infinite.} \end{cases}$$

2. (i) $(a, b) \subset [a, b]$ and (a, b) is open. It follows that

$$(a, b) \subset \text{int}([a, b])$$

so that $\text{int}([a, b])$ is either (a, b) or $[a, b]$. But we already saw in homework 1, 4 (iv) that $[a, b]$ is not open. It follows that

$$\text{int}([a, b]) = (a, b).$$

(ii) We saw that (b, ∞) is open in homework 1, 4 (v). By a similar argument, it follows that $(-\infty, a)$ is open. Indeed

$$(-\infty, a) = \bigcup_{n \in \mathbb{Z}, n < a} (n, a).$$

It follows that the complement

$$[a, b]$$

is closed. As

$$(a, b) \subset [a, b]$$

it follows that

$$\overline{(a, b)} \subset [a, b].$$

But then the closure of $[a, b)$ either $[a, b)$ or $[a, b]$.

We check that $[a, b)$ is not closed. The complement of $[a, b)$ is $(-\infty, a) \cup [b, \infty)$. If we intersect this with the open set

$$(a, b + 1)$$

then we get the set $[b, b + 1)$, which is not open. Thus $(-\infty, a) \cup [b, \infty)$ is not open. It follows that $[a, b)$ is not closed. Hence

$$\overline{[a, b)} = [a, b].$$

(iii)

$$\begin{aligned}\partial[a, b) &= \overline{[a, b)} \setminus \text{int}([a, b)) \\ &= [a, b] \setminus (a, b) \\ &= \{a, b\}.\end{aligned}$$

3. Let C be the complement of B . We have to show that C is open. Pick $c \in C$. Then $c \notin B$ and so

$$\rho = d(a, c) > r.$$

Let $t = \rho - r$ and let

$$U = B_t(c),$$

the open ball of radius t about c . Suppose that $c' \in C$. We have

$$d(c, c') < t \quad \text{and} \quad d(a, c) \leq d(a, c') + d(c', c).$$

It follows that

$$\begin{aligned}d(a, c') &\geq d(a, c) - d(c, c') \\ &> \rho - t \\ &= r.\end{aligned}$$

Thus $c' \in C$. It follows that $c' \in U$ so that $U \subset C$. Therefore C is open.

Hence B is closed.

4. (i) False (despite the fact that $\bar{B}_r(a)$ is called the closed ball and despite the intuition given to us by $\mathbb{R}^n$).

Let $X = \{a, b\}$. Define a metric by the rule

$$d(a, b) = 1.$$

(everything else is dictated by the axioms of a metric). This is the metric that appeared in homework 1, question 3. The resulting topology is the discrete topology.

Consider the open ball of radius 1 about a . Then this contains a but not b ,

$$B_1(a) = \{a\}.$$

The closure of this is the same thing (since every set is closed in the discrete topology).

But the closed ball of radius 1 centred at a contains a and b :

$$\bar{B}_1(a) = \{a, b\}.$$

(ii) True; indeed this is (2) of (4.5).

$\text{int}(Y)$ is open and so $X \setminus \text{int}(Y)$ is closed. $\text{int}(Y) \subset Y$ and so

$$X \setminus Y \subset X \setminus \text{int}(Y).$$

As $X \setminus \text{int}(Y)$ is a closed set containing $X \setminus Y$ it follows that

$$\overline{X \setminus Y} \subset X \setminus \text{int}(Y),$$

that is, the LHS is contained in the RHS.

$\overline{X \setminus Y}$ is closed. Thus

$$X \setminus (\overline{X \setminus Y})$$

is open.

$$X \setminus (\overline{X \setminus Y}) \subset X \setminus (X \setminus Y) = Y.$$

Thus

$$X \setminus (\overline{X \setminus Y}) \subset \text{int}(Y).$$

Taking complements, we see that the RHS is a subset of the LHS. Thus we have equality.

(iii) False. Let $X = \mathbb{R}$, with the Euclidean topology. Let

$$Y = (-1, 0) \cup (0, 1) \quad \text{and} \quad Z = \{0\}.$$

Then Y is open and so

$$\text{int}(Y) = Y.$$

Z is not open and so

$$\text{int}(Z) = \emptyset.$$

Thus

$$\text{int}(Y) \cup \text{int}(Z) = (-1, 0) \cup (0, 1).$$

But

$$Y \cup Z = (-1, 1),$$

which is open and so

$$\begin{aligned} \text{int}(Y \cup Z) &= (-1, 1) \\ &\neq (-1, 0) \cup (0, 1) \\ &= \text{int}(Y) \cup \text{int}(Z). \end{aligned}$$

(iv) False. The complement of the integers is the union of the open intervals

$$(n, n+1),$$

which is open. It follows that $\mathbb{Z}$ is closed. In particular it is far from being dense.

(v) True. We have to show that every open set intersects the rationals. It is enough to show every open interval intersects the rationals. Let $U = (a, b)$ be an open interval. Then we may find a rational number q between a and b , $a < q < b$. But then we may find $q \in (a, b)$ and so $(a, b) \cap \mathbb{Q}$ is non-empty. But then the rationals are dense.

5. This is harder than it might first appear. Let U be an open subset. Define a relation $\sim$ on U by the rule

$$u \sim v \quad \text{if and only if there is an open interval } \{u, v\} \subset (a, b) \subset U.$$

We claim that $\sim$ is an equivalence relation. We have to show that $\sim$ is reflexive, symmetric and transitive.

Suppose that $u \in U$. As U is open we may find $u \in (a, b) \subset U$. But then $u \sim u$ and so $\sim$ is reflexive.

Suppose that u and $v \in U$ and $u \sim v$. Then we may find $(a, b) \subset U$ such that u and $v \in (a, b)$. But then v and $u \in (a, b)$ and $(a, b) \subset U$. Thus $\sim$ is symmetric.

Suppose that u, v and $w \in U$ and $u \sim v, v \sim w$. This implies we may find $(a, b), (c, d) \subset U$ such that u and $v \in (a, b)$ and v and $w \in (c, d)$. Consider $(a, b) \cup (c, d)$. By assumption $v \in (a, b) \cap (c, d)$. It follows that

$$(a, b) \cup (c, d) = (\alpha, \beta)$$

where $\alpha = \min(a, c)$ and $\beta = \max(b, d)$. Clearly u and $w \in (\alpha, \beta)$ and $(\alpha, \beta) \subset U$. But then $u \sim w$. Thus $\sim$ is transitive.

It follows that $\sim$ is an equivalence relation. The equivalence classes partition U into disjoint sets. It therefore suffices to show that each equivalence class is an open interval. As each equivalence class is a union of open intervals, each equivalence class is open. There is no harm in assuming that U is an equivalence class.

In this case note that U has the following curious property. If $u < v \in U$ then (since $u \sim v$) $[u, v] \subset (a, b) \subset U$

Let α be the infimum of the elements of U and let β be the supremum. I claim that α is not in U . If it were, then by the observation above, there would be smaller elements of U , a contradiction. It follows that $\alpha \notin U$. Similarly $\beta \notin U$.

Suppose that $r \in (\alpha, \beta)$. By definition of α we may find $u \in U$ and $u < r$. Similarly by definition of β , we may find $v \in U$, $v > r$. Since $u \sim v$ it follows that $(u, v) \subset U$. In particular $r \in U$. Thus

$$U = (\alpha, \beta).$$

6. (i) 14.

Challenge Problems: (Just for fun)

(ii) We have to exhibit an example where the maximum 14 is achieved and we have to show that there are never more than 14. We will do this in reverse order. In practice, probably the best way to solve this problem is to work on both problems simultaneously.

Note that the closure of the closure is simply the closure and the complement of the complement is the original set. So if we make a graph whose vertices are the subsets of X and there is an edge between two vertices if one subset is the closure of the other or one subset is the complement of the other, then the degree of any vertex is at most 2. If we want to bound the number of vertices of a connected component of this graph we need to show that we come back on ourselves.

Suppose A is a closed set. Then the only thing to do is take the complement, $U = X \setminus A$. Then U is open. Let $B = \overline{U}$ be the closure. Then the complement $V = X \setminus B$ is open. Let $C = \overline{V}$ be the closure. Let $W = X \setminus C$ be the complement, another open set and let $D = \overline{W}$ be the closure.

The key result is:

Claim 0.1. $B = D$.

Proof. We have

$$\begin{aligned} D &= \overline{X \setminus C} \\ &= \overline{X \setminus \overline{V}} \\ &= \overline{\text{int}(X \setminus V)} \\ &= \overline{\text{int}(B)} \\ &= B. \end{aligned} \quad \square$$

Thus if we start with a closed set A then we can form at most five other distinct subsets, U , V , W , B and C .

Now suppose that Y_1 is an arbitrary subset. If we take the closure of Y_1 to get Y_2 then we at get at most 5 other subsets, Y_3 , Y_4 , Y_5 , Y_6 and Y_7 . Similarly if Z_1 is the complement of Y_1 , then the only thing to do is take the closure of Z_1 to get six other subsets, Z_2 , Z_3 , Z_4 , Z_5 , Z_6 and Z_7 . Thus we can get at most 14 subsets.

Now for an example. We take $X = \mathbb{R}$, with the Euclidean topology. The tricky thing is to make sure that we keep the Y's and Z's distinct. Let

$$Y_1 = [0, 1) \cup (1, 2] \cup \{3\} \cup ([4, 5] \cap \mathbb{Q}).$$

The closure of Y_1 is

$$Y_2 = [0, 2] \cup \{3\} \cup [4, 5]$$

The complement of Y_2 is

$$Y_3 = (-\infty, 0) \cup (2, 3) \cup (3, 4) \cup (5, \infty).$$

The closure of Y_3 is

$$Y_4 = (-\infty, 0] \cup [2, 4] \cup (5, \infty).$$

The complement of Y_4 is

$$Y_5 = (0, 2) \cup (4, 5)$$

The closure of Y_5 is

$$Y_6 = [0, 2] \cup [4, 5]$$

The complement of Y_6 is

$$Y_7 = (-\infty, 0) \cup (2, 4) \cup (5, \infty).$$

The closure of this is Y_4 , as expected.

Now suppose we take the complement of Y_1 to get

$$Z_1 = (-\infty, 0) \cup \{1\} \cup (2, 3) \cup (3, 4) \cup ((4, 5) \cap \mathbb{I}) \cup (5, \infty)$$

The closure of Z_1 is

$$Z_2 = (-\infty, 0] \cup \{1\} \cup [2, \infty).$$

The complement of Z_2 is

$$Z_3 = (0, 1) \cup (1, 2).$$

The closure of Z_3 is

$$Z_4 = [0, 2].$$

The complement of Z_4 is

$$Z_5 = (-\infty, 0) \cup (2, \infty).$$

The closure of Z_5 is

$$Z_6 = (-\infty, 0] \cup [2, \infty).$$

The complement of Z_6 is

$$Z_7 = (0, 2).$$

The closure of this is Z_4 , as expected.

7. (a) We show that the subvarieties have the correct properties of closed sets. The equation $1 = 0$ is never satisfied (and 1 is a polynomial) and so $\emptyset$ is a subvariety. The equation $0 = 0$ is always satisfied (and 0 is a polynomial) and so X is a subvariety.

Suppose that V_i are subvarieties of k^n . By assumption we are given sets of polynomials S_i , so that $V_i = V(S_i)$. Let

$$S = \bigcup_i S_i.$$

Then we have

$$\begin{aligned} V(S) &= V\left(\bigcup_i S_i\right) \\ &= \bigcap_i V(S_i) \\ &= \bigcap_i V_i. \end{aligned}$$

In words, the polynomials in S , the union of all polynomials in each S_i , vanishes on the intersection. Thus the arbitrary intersection of subvarieties is a subvariety.

Now suppose that V_1 and V_2 are two subvarieties. We may suppose that $V = V(S_1)$ and $W = V(S_2)$. Let S be the set of all pairwise products of elements of S_1 and S_2

$$V(S) = \{ f_1 f_2 \mid f_1 \in S_1, f_2 \in S_2 \}.$$

Note that

$$V(f_1 f_2) = V(f_1) \cup V(f_2).$$

In words, the polynomial $f_1 f_2$ vanishes where either factor vanishes. Then

$$\begin{aligned} V(S) &= \bigcap_{f_1 \in S_1, f_2 \in S_2} V(f_1 f_2) \\ &= \bigcap_{f_1 \in S_1} V(f_1) \cup \bigcap_{f_2 \in S_2} V(f_2) \\ &= V(S_1) \cup V(S_2). \end{aligned}$$

Thus the union of two subvarieties is a subvariety. Thus we get a topology where the subvarieties are the closed subsets.

(b) Now we are looking at polynomials in one variable $f(x)$. Any non-zero polynomial has at most finitely many zeroes (at most the degree of f). Thus the closed sets are the finite sets, plus k .

(c) U_f is the complement of $V(f)$. Thus U_f is open.

(d) Suppose that U is an open subset. Then the complement of U is a subvariety V . Suppose that V is defined by the polynomials S . It follows that

$$U = \{ (a_1, a_2, \dots, a_n) \in k^n \mid \text{we may find } f \in S \text{ such that } f(a_1, a_2, \dots, a_n) \neq 0 \}.$$

Pick $p = (a_1, a_2, \dots, a_n) \in k^n$. If $p \in U$ then we may find $f \in S$ such that $f(p) \neq 0$. But then

$$p \in U_f \subset U.$$

It follows that U_f is a basis for the Zariski topology.

(e) Let $V = V(S)$. Let $I = \langle S \rangle$ be the ideal generated by S . Note that the sum of two polynomials vanishing on V , vanishes on V . Similarly if f vanishes on V and g is any other polynomial, then fg vanishes on V . Thus

$$\begin{aligned} V(I) &\subset V(S) \\ &= V \\ &\subset V(I). \end{aligned}$$

Thus $V = V(I)$. Hilbert's basis theorem implies that the ideal I is finitely generated. Suppose that $f_1, f_2, \dots, f_k$ are polynomials such that

$$I = \langle f_1, f_2, \dots, f_k \rangle.$$

Note that

$$\begin{aligned} V &= V(I) \\ &= V(f_1, f_2, \dots, f_k). \end{aligned}$$

(f) Let U be an open subset. The complement V is a closed subset. By (e)

$$V = V(f_1, f_2, \dots, f_k),$$

for some polynomials $f_1, f_2, \dots, f_k$. In this case

$$U = \bigcup_{i=1}^k U_{f_i}.$$