

MODEL ANSWERS TO THE FIRST HOMEWORK

1. There are quite a few of these, so we will have to be systematic about this. Let us consider the number x of open subsets with one element and the number y of open subsets with two elements. Note that $0 \leq x \leq 3$ and $0 \leq y \leq 3$. If there are two open sets with one element then their union is an open subset with two elements. Thus if $x \geq 2$ then $y \geq 1$. If there are two open sets with two elements then their intersection is an open subset with one element. Thus if $y \geq 2$ then $x \geq 1$.

We now consider cases. If $x = 3$ then we get all subsets with one element and so every subset is open. Thus we get the discrete topology. In this case $y = 3$. Similarly if $y = 3$ then every set with two elements is open and we get the discrete topology.

Concisely we can say that

$$|x - y| \leq 1 \quad \text{and} \quad x = 3 \quad \text{if and only if} \quad y = 3.$$

If $x = 0$ then there are no open subsets with one element. Then $y \leq 1$. If $y = 0$ then we have the trivial topology. If $x = 0$ and $y = 1$ then $\mathcal{T}$ is one of

$$\{X, \{b, c\}, \emptyset\} \quad \{X, \{a, c\}, \emptyset\} \quad \text{or} \quad \{X, \{a, b\}, \emptyset\}.$$

Now suppose that $x = 1$. Then $0 \leq y \leq 2$. If $y = 0$ then $\mathcal{T}$ is one of

$$\{X, \{a\}, \emptyset\} \quad \{X, \{b\}, \emptyset\} \quad \text{or} \quad \{X, \{c\}, \emptyset\}.$$

Before we proceed further, it might be worthwhile to observe that the axioms of topology are symmetric when X is finite, since all infinite unions reduce to a finite union. Thus the operation of taking complements interchanges two topologies (the open sets in one topology become the closed subsets in the other topology). For example, the two series of examples above are related by taking complements.

If $x = y = 1$ then $\mathcal{T}$ is one of

$$\{X, \{a, b\}, \{a\}, \emptyset\} \quad \text{or} \quad \{X, \{a, b\}, \{b\}, \emptyset\}$$

or what you get by applying (once or twice) the symmetry $a \longrightarrow b \longrightarrow c \longrightarrow a$.

If $x = 1$ and $y = 2$ then $\mathcal{T}$ is

$$\{X, \{a, b\}, \{a, c\}, \{a\}, \emptyset\}$$

or what you get by applying the symmetry above.

Now suppose that $x = 2$, so that there are two open sets with one element. Then $1 \leq y \leq 2$. If $x = 2$ and $y = 1$ then $\mathcal{T}$ is

$$\{X, \{a, b\}, \{a\}, \{b\}, \emptyset\}$$

or what you get by applying the symmetry above. Note that the last two examples are related by taking complements.

If $x = 2$ and $y = 2$ then $\mathcal{T}$ is

$$\{X, \{a, b\}, \{a, c\}, \{a\}, \{b\}, \emptyset\}$$

or what you get by any symmetry of the alphabet $\{a, b, c\}$.

2. Let

$$\mathcal{T} = \{X, \{a, b\}, \{c, d\}, \emptyset\}.$$

It is easy to check that this is a topology. It is clearly neither the trivial topology nor the discrete topology.

The open set

$$\{a, b\}$$

is closed as its complement

$$\{c, d\}$$

is open. It is then easy to see that a subset is open if and only if it is closed.

3. (i) It is clear that $d(x, y) = 0$ if and only if $x = y$, so that the first axiom of a metric holds. It is also clear that $d(x, y) = d(y, x)$, so that the second axiom holds. We check the triangle inequality. Suppose we are given x, y and z . We want to check that

$$d(x, z) \leq d(x, y) + d(y, z).$$

There are two cases. If $x = z$ then the LHS is zero and the RHS is obviously non-negative and so the inequality holds in this case. If $x \neq z$ then either $x \neq y$ or $y \neq z$. But then the RHS is at least one. As the LHS is one the inequality holds in this case as well.

Thus we have a metric.

(ii) The open ball of radius one about a contains only a ,

$$B_1(a) = \{a\}.$$

Thus every set with one element is open. But then every set is open and we have the discrete topology.

4. (i) False. For example, let $X = \{a, b\}$ and let

$$\mathcal{T}_1 = \{X, \{a\}, \emptyset\} \quad \text{and} \quad \mathcal{T}_2 = \{X, \{b\}, \emptyset\}.$$

Then $\{a\}$ is an open subset in the first topology but not the second, whilst $\{b\}$ is an open subset in the second topology but not the first. Thus we cannot compare $\mathcal{T}_1$ and $\mathcal{T}_2$.

(ii) False (but only just). Let $X = \emptyset$. This is a metric space, with the trivial metric. In fact there is only one function on the product of the empty set with itself, the empty function. It satisfies the axioms for a metric in a vacuous way. There is only one topology on the empty set, so this metric space must give us this topology. This topology is also the trivial topology.

(iii) False. For example, suppose that $X = \{a, b\}$. Define the functions

$$d_i: X \times X \longrightarrow \mathbb{R}$$

$i = 1$ and 2 by the rule

$$d_i(x, y) = \begin{cases} 0 & \text{if } x = y \\ i & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} d_1(a, b) &= 1 \\ &\neq 2 \\ &= d_2(a, b). \end{aligned}$$

Thus $d_1 \neq d_2$. We already saw that d_1 is a metric and the associated topology is the discrete topology. The same argument show that d_2 is a metric and the associated topology is the discrete topology.

(iv) False. Consider the point $a \in [a, b)$. If $[a, b)$ were open then we could find a small open ball $U = B_r(a)$ about a such that $U \subset [a, b)$. If we put $c = a - r/2$ then

$$c \in U \quad \text{whilst} \quad c \notin [a, b).$$

Thus $[a, b)$ is not open.

(v) True. As

$$(a, \infty) = \bigcup_n (a, a + n)$$

it follows that (a, ∞) is open in the Euclidean topology.

(vi) False. Let $X = \mathbb{N}$. Then the set of even numbers and the set of prime numbers are both infinite sets. The intersection is the set $\{2\}$, as 2 is the only even prime. This is a finite set, so infinite sets are not closed under intersection and we don't have a topology.

(vii) True. Both topologies are defined by bases, the first by open intervals, the second by open balls. In fact these are the same. If (a, b) is an open interval then

$$(a, b) = B_r(c),$$

where

$$c = \frac{a+b}{2} \quad \text{and} \quad r = \frac{a-b}{2}.$$

Vice-versa, if $B_r(a)$ is an open ball then

$$B_r(a) = (a-r, a+r).$$

5. (i) $(-\infty, \infty) = \mathbb{R}$ and $(\infty, \infty) = \emptyset$. Thus $\emptyset$ and $X \in \mathcal{T}$. Suppose that we are given elements $A_1, A_2, \dots$ of $\mathcal{T}$. Then $A_i = (a_i, \infty)$ for some a_i . We want to show that the union is an element of $\mathcal{T}$. Let $a = \inf_i a_i$ (this exists by standard results about the real numbers $\mathbb{R}$). We claim

$$\begin{aligned} \bigcup_i A_i &= \bigcup_i (a_i, \infty) \\ &= (a, \infty). \end{aligned}$$

Indeed if $b \in (a_i, \infty)$ then

$$\begin{aligned} b &> a_i \\ &\geq a, \end{aligned}$$

so that $b \in (a, \infty)$. Thus the LHS is contained in the RHS. Now suppose that $b \in (a, \infty)$. Then $b > a$. As a is an infimum, it follows that there is an index i such that $a \leq a_i < b$. But then $b \in (a_i, \infty)$ and so the RHS is contained in the LHS. But then we have equality. Thus $\mathcal{T}$ is closed under arbitrary union.

Finally suppose that U and $V \in \mathcal{T}$. Then $U = (a, \infty)$ and $V = (b, \infty)$. Let $c = \max(a, b)$. Then

$$U \cap V = (c, \infty).$$

Thus $\mathcal{T}$ is closed under intersection and so $\mathcal{T}$ is a topology.

(ii) We already saw that

$$(a, \infty)$$

is open in the the Euclidean topology in 4 (v). Thus the Euclidean topology is finer than the topology $\mathcal{T}$.

Challenge Problems: (Just for fun)

6. $A \subset A$ and so $\leq$ is reflexive. If $A \subset B$ and $B \subset A$ then $A = B$ and so $\leq$ is anti-symmetric. If $A \subset B$ and $B \subset C$ then $A \subset C$ and so $\leq$ is transitive. Thus $(L, \leq)$ is a partially ordered set.

Let $W \subset L$ be a collection of subsets A_i . Let

$$U = \bigcup_4 A_i$$

be the union. Then $A_i \subset U$ so that $A_i \leq U$. Now suppose that $B \subset X$ has the property that $A_i \leq B$. Then $A_i \subset B$ so that $U \subset B$. Thus

$$U = \bigvee W.$$

is the supremum.

Now let

$$L = \bigcap_i A_i$$

be the intersection. Arguing as above, it follows that

$$L = \bigwedge W$$

is the infimum.

7. (a) If $a = f(0_L)$ then

$$\begin{aligned} 0_L &\leq a \\ &= f(0_L) \end{aligned}$$

where the first inequality holds as 0_L is the smallest element of L . Thus $0_L \in D$.

(b) If $a \in D$ then $a \leq u$. Applying f we get

$$\begin{aligned} a &\leq f(a) \\ &\leq f(u). \end{aligned}$$

As u is the supremum, it follows that $u \leq f(u)$. But then $u \in D$. If we start with the inequality $u \leq f(u)$ and apply f then we get

$$f(u) \leq f(f(u)).$$

It follows that $f(u) \in D$. But then $f(u) \leq u$. It follows that $u = f(u)$ and so $u \in P$.

(c) As u is bigger than any element of D it is certainly bigger than any element of P . Thus $u = 1_P$ is the largest element of P .

(d) It is clear that $(L, \preceq)$ is a complete lattice and that f is monotonic on this lattice. By what we already proved the set of fixed points P has a biggest element with respect to $\preceq$. But the biggest element with respect to $\preceq$ is the smallest element with respect to $\leq$.

(e) If $W \subset [w, 1_L]$ then both $\bigvee W$ and $\bigwedge W \geq w$ so that $\bigvee W$ and $\bigwedge W \in [w, 1_L]$. Thus $([w, 1_L], \leq)$ is a complete lattice. As f is monotonic, it follows that $f([w, 1_L]) \subset [w, 1_L]$.

Note that the set of fixed points of f restricted to $[w, 1_L]$ is $P \cap [w, 1_L]$. By what we already proved this has a smallest element, $0_{P \cap [w, 1_L]}$. It is easy to see that this is the supremum of W inside P .

8. (a) If $A \leq B$ then $A \subset B$ and so

$$f(A) \subset f(B) \quad \text{which implies that} \quad Y \setminus f(B) \subset Y \setminus f(A).$$

It follows that

$$g(Y \setminus f(B)) \subset g(Y \setminus f(A)).$$

But then

$$\begin{aligned}\phi(A) &= X \setminus g(Y \setminus f(A)) \\ &\subset X \setminus g(Y \setminus f(B)) \\ &= \phi(B),\end{aligned}$$

so that ϕ is monotonic.

(b) The Knaster-Tarski theorem implies that ϕ has a fixed point C , a subset $C \subset X$ such that

$$\begin{aligned}C &= \phi(C) \\ &= X \setminus g(Y \setminus f(A)).\end{aligned}$$

Taking the complement of both sides we get

$$X \setminus C = g(Y \setminus f(C)).$$

(c) The function $f|_C: C \rightarrow Y$ is injective and so the function

$$f|_C: C \rightarrow f(C)$$

is bijective. On the other hand the function

$$g|_{Y \setminus f(C)}: Y \setminus f(C) \rightarrow X$$

is injective and so

$$g|_{Y \setminus f(C)}: Y \setminus f(C) \rightarrow X \setminus C$$

is bijective. In particular the inverse function exists. It follows that if we define the function

$$h: X \rightarrow Y$$

by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in C \\ g^{-1}(x) & \text{if } x \in X \setminus C \end{cases}$$

then h is a bijection.