

SECOND MIDTERM
MATH 190A, UCSD, WINTER 25

You have 50 minutes.

There are 3 problems, and the total number of points is 65. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:_____

Signature:_____

Student ID #:_____

Problem	Points	Score
1	15	
2	30	
3	20	
4	10	
5	10	
Total	65	

1. (15pts) *Give the definition of a continuous function.*

A function $f: X \longrightarrow Y$ between two topological spaces is continuous if the inverse image of every open set is open.

- (ii) *Give the definition of a disconnected topological space.*

The topological space X is disconnected if there are two non-empty open sets U and V such that $X = U \cup V$ and $U \cap V = \emptyset$.

- (iii) *Give the definition of a compact topological space.*

A topological space X is compact if every open cover has a finite sub-cover.

2. (30pts) *True or False? If true then prove the result and if false then give a counterexample.*

(i) *The composition of continuous functions is continuous.*

True. Let

$$f: X \longrightarrow Y \quad \text{and} \quad g: Y \longrightarrow Z$$

be two continuous functions. Let

$$h = g \circ f: X \longrightarrow Z$$

be the composition. Let $W \subset Z$ be an open set. Let $V = g^{-1}(W)$ and $U = f^{-1}(V)$. Then V is open, as g is continuous, and so U is open, as f is continuous. We have

$$\begin{aligned} h^{-1}(W) &= (g \circ f)^{-1}(W) \\ &= f^{-1}(g^{-1}(W)) \\ &= f^{-1}(V) \\ &= U. \end{aligned}$$

Thus h is continuous.

(ii) *Any interval in $\mathbb{R}$ is connected.*

True. Suppose that $[0, 1]$ is the disjoint union of two open subsets U and V . Suppose U is non-empty. Let l be the infimum of the elements of U .

Suppose that $l \in V$. Then $l < 1$, as U is non-empty. As V is open, we may find $b > l$ such that

$$[l, b) \subset V.$$

In this case $[l, b) \cap U = \emptyset$ so that all of the elements of U are at least b . In this case $b > l$ is a bigger lower bound than l . As this is not possible, we must have $l \in U$.

Suppose that $l > 0$. As U is open this implies that

$$(a, l] \subset U,$$

for some $a < l$. Let

$$c = a + \frac{l - a}{2}.$$

Then $c \in U$ and $c < l$. As this is not possible, we must have $l = 0$.

Note that we just proved the following: if U is non-empty then it must contain 0. By the same token, if V were non-empty, it would follow that $0 \in V$. As this is not possible, it follows that V is empty. But then $[0, 1]$ is connected.

(iii) $[0, 1)$ and $(0, 1)$ are homeomorphic.

False. Suppose that

$$f: X \longrightarrow Y$$

is a homeomorphism. Let $x \in X$ be a point and let $y = f(x) \in Y$. Then f induces a homeomorphism between $X \setminus \{x\}$ and $Y \setminus \{y\}$. In particular, $X \setminus \{x\}$ is connected if and only if $Y \setminus \{y\}$ is connected. Consider what happens if we remove a point y from $(0, 1)$. Then we get two intervals, $(0, y)$ and $(y, 1)$, so that $(0, 1) \setminus \{y\}$ is never connected. But if we remove the point $x = 0$ from $[0, 1)$ then we get $(0, 1)$, which is connected.

Thus $X = [0, 1)$ and $Y = (0, 1)$ are not homeomorphic.

3. (20pts) (i) Let X , Y and Z be three topological spaces. Show that

$$f: Z \longrightarrow X \times Y$$

is continuous if and only if the two maps

$$p \circ f: Z \longrightarrow X \quad \text{and} \quad q \circ f: Z \longrightarrow Y$$

are continuous, where $p: X \times Y \longrightarrow X$ and $q: X \times Y \longrightarrow Y$ are the two natural projections.

We first check that p is continuous. If U is open in X then $p^{-1}(U) = U \times Y$, which is open in $X \times Y$. Thus p is continuous. Similarly q is continuous.

Now suppose that f is continuous. Then $p \circ f$ and $q \circ f$ are continuous, as the composition of continuous functions is continuous.

Conversely, suppose that $f_1 = p \circ f$ and $f_2 = q \circ f$ are continuous. We have

$$f = (f_1, f_2).$$

Suppose that U is open in X and that V is open in Y . Then

$$f_1^{-1}(U) \quad \text{and} \quad f_2^{-1}(V)$$

are open as f_1 and f_2 are continuous. Therefore

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$$

is open, as it is the intersection of open sets.

As sets of the form $U \times V$ are a basis for the product topology, it follows that f is continuous.

(iv) *Show that the product of two path-connected spaces is path-connected.*

True. Suppose that X and Y are path-connected and let

$$(x_i, y_i) \in X \times Y \quad \text{for} \quad i = 0, 1.$$

By assumption there are paths

$$f: [0, 1] \longrightarrow X \quad \text{and} \quad g: [0, 1] \longrightarrow Y$$

from x_0 to x_1 and from y_0 to y_1 , so that $f(i) = x_i$ and $g(i) = y_i$, $i = 0$ and 1 . Consider the function

$$h: [0, 1] \longrightarrow X \times Y$$

given by

$$h(t) = (f(t), g(t)).$$

Then $f = p \circ h$ and $g = q \circ h$ are continuous, so that h is continuous.

But

$$h(i) = (x_i, y_i) \quad \text{for} \quad i = 0, 1,$$

so that h is a path from (x_0, y_0) to (x_1, y_1) . But then $X \times Y$ is path-connected.

Bonus Challenge Problems

4. (10pts) *Give the construction of the Cantor set and show that it is everywhere disconnected, that it is closed and that every point is an accumulation point.*

We define a sequence of closed subsets C_n by recursion. C_n will be the union of 2^n closed intervals of length 3^{-n} .

We put $C_0 = [0, 1]$. This is $1 = 2^0$ interval of length $1 = 3^0$. Suppose we have defined C_{n-1} . We may suppose by induction that C_{n-1} is the union of 2^{n-1} closed intervals of length 3^{1-n} . We then remove the middle third open interval from each interval of C_{n-1} . In this way we get $2 \cdot 2^{n-1} = 2^n$ intervals each of length $3^{-1}3^{1-n} = 3^{-n}$.

The Cantor set is then the intersection,

$$C = \bigcap_n C_n.$$

The Cantor set is closed, as it is the intersection of closed sets.

Let I be an interval contained in C . Then I is contained in C_n and so its length is no more than 3^{-n} . It follows that the length of I is zero, so that I is a single point. Thus C is everywhere disconnected.

Pick a point r of C . Then r belongs to C_n , so that it belongs to one of 2^n intervals. Pick an endpoint r_n of this interval (other than r). Then

$$|r_n - r| \leq \frac{1}{3^n}.$$

Thus r is an accumulation point of

$$\{r_n \mid n \in \mathbb{N}\}.$$

Hence every point of C is an accumulation point.

5. (10pts) Let $(X, \mathcal{T})$ be the topological space, where $X = \mathbb{R}$ and a basis $\mathcal{B}$ for $\mathcal{T}$ is

$$\mathcal{B} = \{ [a, b) \mid a < b \in \mathbb{R} \}.$$

Show that $[0, 1]$ is not compact.

First note that

$$\{ 1 \} = [0, 1] \cap [1, 2)$$

is open in $[0, 1]$.

Let

$$\mathcal{U} = \{ [0, 1 - \frac{1}{n}) \mid n \in \mathbb{N} \} \cup \{ 1 \}.$$

This is an open cover of $[0, 1]$ with no finite subcover.