

FIRST MIDTERM
MATH 190A, UCSD, WINTER 25

You have 50 minutes.

There are 3 problems, and the total number of points is 65. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:_____

Signature:_____

Student ID #:_____

Problem	Points	Score
1	15	
2	20	
3	30	
4	10	
5	10	
Total	65	

1. (15pts) *Give the definition of a topological space.*

A topological space consists of a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X .

(ii) *Give the definition of the closure of a set.*

If Y is a subset of a topological space X then the closure $\overline{Y}$ is the smallest closed set containing Y .

(iii) *Give the definition of the basis for a topology.*

If $(X, \mathcal{T})$ is a topological space then a basis $\mathcal{B}$ for $\mathcal{T}$ is a subset $\mathcal{B} \subset \mathcal{T}$ such that every set in $\mathcal{T}$ is the union of elements of $\mathcal{B}$.

2. (20pts) Let X be a topological space. Show that a collection $\mathcal{B}$ of subsets of X is the basis for a topology on X if and only if the following two conditions hold:

- (1) For every $x \in X$ there is an element $U \in \mathcal{B}$ such that $x \in U$.
- (2) For every $U_1 \in \mathcal{B}$ and $U_2 \in \mathcal{B}$ and $x \in U_1 \cap U_2$ we may find $U_3 \in \mathcal{B}$ such that

$$x \in U_3 \subset U_1 \cap U_2.$$

Suppose that $\mathcal{B}$ is a basis for the topology $\mathcal{T}$.

As $X \in \mathcal{T}$ is open it follows that X is a union of elements of $\mathcal{B}$, that is, the elements of $\mathcal{B}$ cover X . Thus (1) holds.

Now suppose that U_1 and $U_2 \in \mathcal{B}$. Then U_1 and U_2 are open. It follows that $U_1 \cap U_2$ is open. But then $U_1 \cap U_2$ is the union of elements of $\mathcal{B}$. Thus (2) holds.

Now suppose that (1) and (2) hold. Let $\mathcal{T}$ be the collection of all subsets of X which are the union of elements of $\mathcal{B}$. We show that $\mathcal{T}$ is a topology. We have to check three things.

$\emptyset$ is the union of no elements of $\mathcal{B}$. Thus $\emptyset \in \mathcal{T}$. The union of all elements of $\mathcal{B}$ is the whole of X , by (1). Thus $X \in \mathcal{T}$.

Now suppose that U_i , $i \in I$, is a collection of elements of $\mathcal{T}$. By assumption, we may find subsets $B_{i\alpha} \in \mathcal{B}$ such that U_i is a union of $B_{i\alpha}$,

$$U_i = \bigcup_{\alpha \in I_i} B_{i\alpha}.$$

In this case

$$\begin{aligned} U &= \bigcup_{i \in A} U_i \\ &= \bigcup_{i \in A} \bigcup_{\alpha \in I_i} B_{i\alpha} \\ &= \bigcup_{i \in A, \alpha \in I_i} B_{i\alpha} \in \mathcal{T}. \end{aligned}$$

Thus $\mathcal{T}$ is closed under arbitrary unions.

Now suppose that U and $V \in \mathcal{T}$. Then we may find U_α and $V_\beta \in \mathcal{B}$, such that

$$U = \bigcup_{\alpha} U_\alpha \quad \text{and} \quad V = \bigcup_{\beta} V_\beta.$$

Then $U_\alpha \cap V_\beta$ is a union of elements of $\mathcal{B}$, as we are assuming (2). Thus $U_\alpha \cap V_\beta \in \mathcal{T}$. But then

$$\begin{aligned} U \cap V &= \left(\bigcup_{\alpha} U_\alpha \right) \cap \left(\bigcup_{\beta} V_\beta \right) \\ &= \bigcup_{\alpha, \beta} (U_\alpha \cap V_\beta) \in \mathcal{T}, \end{aligned}$$

by what we already proved. Thus $\mathcal{T}$ is closed under intersection. It follows that $\mathcal{T}$ is a topology.

(ii) Now let (X, d) be a metric space. Show that the open balls $B_r(a)$ of radius r , centred at a , form the basis $\mathcal{B}$ of a topology on X ,

$$\mathcal{B} = \{ B_r(a) \mid r > 0, a \in X \}.$$

Suppose that $a \in X$. The $a \in B_1(a)$, so that (1) holds.

Now suppose that we have two open balls $B_r(a)$, $B_s(b)$ and that $c \in B_r(a) \cap B_s(b)$. Let $\rho = d(a, c)$. Then $\rho < r$, as $c \in B_r(a)$. Let $\tau = r - \rho$. Suppose that $x \in B_\tau(c)$. Then

$$\begin{aligned} d(a, x) &\leq d(a, c) + d(c, x) \\ &< \rho + \tau \\ &= \rho + (r - \rho) \\ &= r. \end{aligned}$$

Thus $x \in B_r(a)$. It follows that

$$B_\tau(c) \subset B_r(a).$$

By symmetry

$$B_{\tau'}(c) \subset B_s(b),$$

where $\tau' = s - d(b, c) > 0$. If $t = \min(\tau, \tau')$ then

$$B_t(c) \subset B_r(a) \cap B_s(b).$$

This is (2) and so $\mathcal{B}$ is the basis for a topology on X .

3. (30pts) *True or False? If true then prove the result and if false then give a counterexample.*

(i) *If (X, d) is a metric space, $a \in X$ and $r > 0$ then the closure of the open ball $B_r(a)$ is the closed ball $\bar{B}_r(a)$*

False.

Let $X = \{a, b\}$. Define a metric by the rule

$$d(a, b) = 1.$$

(everything else is dictated by the axioms of a metric). It is straightforward to check that this is a metric. The resulting topology is the discrete topology.

Consider the open ball of radius 1 about a . Then this contains a but not b ,

$$B_1(a) = \{a\}.$$

The closure of this is the same thing (since every set is closed in the discrete topology).

But the closed ball of radius 1 centred at a contains a and b :

$$\bar{B}_1(a) = \{a, b\}.$$

(ii) *The rationals $\mathbb{Q}$ are dense in the reals $\mathbb{R}$.*

True. We have to show that every open set intersects the rationals. It is enough to show that every open interval intersects the rationals. Let $U = (a, b)$ be an open interval. Then we may find a rational number q between a and b , $a < q < b$. But then we may find $q \in (a, b)$ and so $(a, b) \cap \mathbb{Q}$ is non-empty. But then the rationals are dense.

(iii) $(0, 1)$ is homeomorphic to $(0, \infty)$.

Let

$$f: (0, 1) \longrightarrow (0, \infty)$$

be given by the rule

$$x \longrightarrow \frac{x}{1-x}.$$

It is clear that

$$y = \frac{x}{1-x} \in (0, \infty).$$

On the other hand, if

$$g: (0, \infty) \longrightarrow (0, 1) \quad \text{given by} \quad g(x) = \frac{x}{x+1}$$

then it is clear that

$$y = \frac{x}{x+1} \in (0, 1).$$

It is easy to check that g is the inverse of f .

A basis for the subspace topology is given by open intervals. It is enough to check that f and g induce a bijection between open intervals. f sends the interval (a, b) to the interval

$$\left(\frac{a}{1-a}, \frac{b}{1-b}\right).$$

Similarly g sends the open interval (a, b) to the interval

$$\left(\frac{a}{a+1}, \frac{b}{b+1}\right).$$

Thus f is a homeomorphism.

Bonus Challenge Problems

4. (10pts) *Show that the Euclidean topology on $\mathbb{R}^2$ and the product topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, are the same.*

Let X be a set with two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, with two bases $\mathcal{B}_1 \subset \mathcal{T}_1$ and $\mathcal{B}_2 \subset \mathcal{T}_2$. We will show that $\mathcal{T}_1$ finer than $\mathcal{T}_2$.

Note that every element of $\mathcal{B}_1$ is covered by elements of $\mathcal{B}_2$, by hypothesis. Thus

$$\mathcal{B}_1 \subset \mathcal{T}_2.$$

As $W \in \mathcal{T}_1$, it is a union of elements of $\mathcal{B}_1$. It is then also a union of elements of $\mathcal{T}_2$. But then $W \in \mathcal{T}_2$.

A basis for the product topology is given by open rectangles and a basis for the Euclidean topology is given by open balls.

Note that given any circle we can find a square with the same centre which sits inside the circle, maximal with this property. It is enough to do this for the unit circle. In this case we want the square with vertices $(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}})$.

Now given any open ball $U = B_r(a)$, centre a , radius r , and any point b of U then we can find an open ball $V = B_t(b)$ of radius t about b such that V is contained in U . In fact if we let

$$\rho = d(a, b) < r \quad \text{then} \quad t = r - \rho > 0$$

works, by a simple application of the triangle inequality.

Note take the biggest square, centred at b , inside the circle of radius t about b . Then every point of the open square is contained in V , which is contained in U .

We have shown that the product topology is finer than the Euclidean topology.

To show the reverse, start with the observation that given any square we can find a circle inside the square with the same centre. Indeed, we may assume that the square has centre the origin and the sides have length 2, so that the vertices are at $(\pm 1, \pm 1)$. Then the circle we are looking for is the unit circle.

Now observe that given any open rectangle and a point of this rectangle, we can find an open square centred at this point, contained in this rectangle. Now put a circle inside the rectangle, with the same centre. Then the open ball with this circle as boundary is contained in the open rectangle.

This shows that the Euclidean topology is finer than the product topology. But then the open sets are the same.

5. (10pts) *Show that one cannot find uncountably many pairwise disjoint subsets of $\mathbb{R}^2$ with non-empty interior.*

Suppose that we can find uncountably many pairwise disjoint subsets of $\mathbb{R}^2$, with non-empty interior. Replacing those subsets by their interior we may assume that these subsets are open. Since any open subset is covered by disks, replacing these subsets by discs we may assume that we can find uncountably many discs in $\mathbb{R}^2$.

Suppose that U_α is a collection of disjoint disks. Given a disc U_α , we may find a point

$$(x_\alpha, y_\alpha) \in U_\alpha$$

with rational coordinates. Let Λ be the indexing set for the alphas. We have constructed a function

$$\phi: \Lambda \longrightarrow \mathbb{Q}^2.$$

This sends the label α to the point

$$\phi(\alpha) = (x_\alpha, y_\alpha).$$

By assumption, if $\alpha \neq \beta$ then U_α and U_β are disjoint. In particular

$$(x_\alpha, y_\alpha) \neq (x_\beta, y_\beta).$$

But then

$$\phi(\alpha) \neq \phi(\beta)$$

so that ϕ is injective.

But

$$\mathbb{Q}^2 \subset \mathbb{R}^2$$

is countable. Thus Λ is countable (or possibly finite, even), since it injects into a countable set.