

5. THE SUBSPACE TOPOLOGY

Definition-Lemma 5.1. *Let $(X, \mathcal{T})$ be a topological space.*

Let $Y \subset X$ be a subset of X . Let

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}.$$

*Then $\mathcal{T}_Y$ is a topology on Y , called the **subspace topology**.*

Proof. We have to show that $\mathcal{T}_Y$ satisfies the axioms of a topology. We have

$$\emptyset = \emptyset \cap Y \quad \text{and} \quad Y = X \cap Y,$$

so that $\emptyset$ and $Y \in \mathcal{T}_Y$.

Suppose that $V_i \in \mathcal{T}_Y$. Then by assumption we may find $U_i \in \mathcal{T}$ such that $V_i = U_i \cap Y$. In this case

$$\begin{aligned} \bigcup_i V_i &= \bigcup_i (U_i \cap Y) \\ &= \left(\bigcup_i U_i \right) \cap Y \\ &\in \mathcal{T}_Y, \end{aligned}$$

since

$$\bigcup_i U_i \in \mathcal{T}.$$

Suppose that V_1 and $V_2 \in \mathcal{T}_Y$. Then we may find U_1 and $U_2 \in \mathcal{T}$ such that

$$V_1 = U_1 \cap Y \quad \text{and} \quad V_2 = U_2 \cap Y.$$

In this case

$$\begin{aligned} V_1 \cap V_2 &= (U_1 \cap Y) \cap (U_2 \cap Y) \\ &= (U_1 \cap U_2) \cap Y \\ &\in \mathcal{T}_Y, \end{aligned}$$

since

$$U_1 \cap U_2 \in \mathcal{T}.$$

Thus $\mathcal{T}_Y$ is a topology on Y . □

Remark 5.2. *Strictly speaking the proof we have given contains a small gap.*

The problem is that the indexing set for i might be quite large. On the other hand, for each index i , we are supposed to choose U_i such that $V_i = U_i \cap Y$. There is no a priori reason we know that we can do this (unless we assume the axiom of choice). However for each index i we can let U_i be the biggest open subset of X whose intersection with

Y is V_i . This makes sense as the union of all such sets is open. This avoids making any choices.

In words, a subset of Y is open in the subspace topology if it is the intersection of an open subset of X with Y .

Notice that this gives us many new examples of topological spaces. Simply take any topological space $(X, \mathcal{T})$ and take any subset Y to get a new topological space $(Y, \mathcal{T}_Y)$. For example, take any geometrical object in $\mathbb{R}^n$. Then this immediately inherits a topology by virtue of sitting inside $\mathbb{R}^n$. For example, all spheres are topological spaces.

Lemma 5.3. *Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$ be any subset.*

- (1) *$A \subset Y$ is closed in the subspace topology if and only if A is the intersection of Y with a closed subset B of X , $A = B \cap Y$.*
- (2) *If $\mathcal{B}$ is a basis for the topology on X and*

$$\mathcal{C} = \{ B \cap Y \mid B \in \mathcal{B} \}$$

then $\mathcal{C}$ is a basis for the subspace topology on Y .

- (3) *If Y is open then $V \subset Y$ is open in the subspace topology if and only if V is open in X .*
- (4) *If Y is closed then $B \subset Y$ is closed in the subspace topology if and only if B is closed in X .*

Proof. Suppose that $B = A \cap Y$, where A is closed in X . Let $U = X \setminus A$. Then U is open in X . Let $V = U \cap Y$. Then V is open in the subspace topology on Y . We have

$$\begin{aligned} Y \setminus V &= Y \setminus (U \cap Y) \\ &= (X \setminus U) \cap Y \\ &= A \cap Y \\ &= B. \end{aligned}$$

Thus B is closed in the subspace topology on Y . The reverse direction is similar. This is (1).

We check that every open subset V of Y is the union of elements of $\mathcal{C}$. As V is open in the subspace topology, we may find U open in X such that $V = U \cap Y$. As $\mathcal{B}$ is a basis for $\mathcal{T}$ it follows that we may find

$U_i \in \mathcal{B}$ covering U . Let $V_i = U_i \cap Y \in \mathcal{C}$. Then

$$\begin{aligned} V &= U \cap Y \\ &= \left(\bigcup_i U_i \right) \cap Y \\ &= \bigcup_i (U_i \cap Y) \\ &= \bigcup_i V_i. \end{aligned}$$

This is (2).

If Y is any subset and $U \subset Y$ is open then $U = U \cap Y$ is surely open in the subspace topology. Now suppose that Y is open and that V is open in the subspace topology. By assumption we may then find $U \subset X$ open such that $V = U \cap Y$. But then V is open in X as it is the intersection of two open subsets. This is (3).

(4) is similar to (3) and is left as an exercise for the reader. $\square$

Example 5.4. Let $X = \mathbb{R}^2$, with the Euclidean topology.

Let

$$Y = \{ (x, 0) \mid x \in \mathbb{R} \}$$

by the x -axis. Then Y is a copy of $\mathbb{R}$, in the obvious way. Y inherits a topology from X .

A basis for the topology on $\mathbb{R}^2$ is given by open balls. These restrict to open intervals on the x -axis. This is the standard basis for the Euclidean topology on the reals. Thus we get the Euclidean topology.

We can generalise this.

Definition-Lemma 5.5. Let (X, d) be a metric space and let $Y \subset X$ be a subset.

Then $(Y, d|_{Y \times Y})$ is a metric space. Moreover the subspace topology and the topology given by this metric are the same.

To prove (5.5) we need to compare two bases on the same set:

Lemma 5.6. Let X be a set with two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, with two bases $\mathcal{B}_1 \subset \mathcal{T}_1$ and $\mathcal{B}_2 \subset \mathcal{T}_2$.

Suppose that every time we are given $x \in X$ and $U_1 \in \mathcal{B}_1$ such that $x \in U_1$ then we can find $U_2 \in \mathcal{B}_2$ such that

$$x \in U_2 \subset U_1.$$

Then $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.

Proof. We have to show that if $W \in \mathcal{T}_1$ then $W \in \mathcal{T}_2$.

Note that every element of $\mathcal{B}_1$ is covered by elements of $\mathcal{B}_2$, by hypothesis. Thus

$$\mathcal{B}_1 \subset \mathcal{T}_2.$$

As $W \in \mathcal{T}_1$, it is a union of elements of $\mathcal{B}_1$. It is then also a union of elements of $\mathcal{T}_2$. But then $W \in \mathcal{T}_2$. $\square$

Proof of (5.5). It almost completely immediate that $d|_{Y \times Y}$ is a metric on Y .

A basis for the topology on X is given by open balls in X ,

$$B_r(a) = \{x \in X \mid d(a, x) < r\}.$$

Note that

$$\begin{aligned} B_r(a) \cap Y &= \{y \in Y \mid d(a, y) < r\} \\ &= \{y \in Y \mid d|_{Y \times Y}(a, y) < r\} \\ &= C_r(a). \end{aligned}$$

Note that it is not correct to say that this is an open ball in Y , simply because a is not necessarily an element of Y .

If $a \in Y$ then $C_r(a)$ is the open ball of radius r centred at a . Thus $C_r(a)$ is open in the topology given by the metric $d|_{Y \times Y}$. Thus the subspace topology is finer than the topology on Y given by the metric.

Suppose that $b \in C_r(a)$. Then $b \in Y \subset X$. Let $s = d(a, b)$. Let $t = r - s > 0$. Then the triangle inequality implies that

$$b \in C_t(b) \subset C_r(a).$$

On the other hand, we have already seen that $C_t(b)$ is an open ball in the topology given by the metric $d|_{Y \times Y}$. Thus the topology given by the metric is finer than the subspace topology. Thus the two topologies coincide. $\square$