

4. BOUNDARY, CLOSURE, DENSITY, INTERIOR AND NEIGHBOURHOOD

Definition-Lemma 4.1. *Let $(X, \mathcal{T})$ be a topological space.*

*Let $Y \subset X$ be a subset of X . The **interior** of Y , denoted $\text{int}(Y)$, is the largest open subset U of X contained in Y .*

Proof. The only thing we have to check is that there is a largest open subset of Y . But this is clear, since the union of all subsets of Y is an open subset of X contained in Y that contains all open subsets of X contained in Y . $\square$

Here are some simple results about the interior:

Lemma 4.2. *Let $(X, \mathcal{T})$ be a topological space and let Y and Z be two subsets of X .*

(1) *Y is open if and only if $\text{int}(Y) = Y$.*

(2) *$\text{int}(X) = X$ and $\text{int}(\emptyset) = \emptyset$.*

(3) *If $Y \subset Z$ then $\text{int}(Y) \subset \text{int}(Z)$.*

(4)

$$\text{int}(Y \cap Z) = \text{int}(Y) \cap \text{int}(Z).$$

Proof. (1) is clear and (1) implies (2).

We have

$$\text{int}(Z) \subset Z \subset Y.$$

Thus $\text{int}(Z)$ is an open subset of X contained in Y . As $\text{int}(Y)$ is the largest open subset of X contained in Y , (3) follows.

$Y \cap Z \subset Z$ and so (3) implies $\text{int}(Y \cap Z) \subset \text{int}(Z)$. Similarly $\text{int}(Y \cap Z) \subset \text{int}(Y)$. Thus the LHS of (4) is contained in the RHS. On the other hand, $\text{int}(Y) \cap \text{int}(Z)$ is the intersection of two open subsets and so $\text{int}(Y) \cap \text{int}(Z)$ is an open subset of X . $\text{int}(Y) \subset Y$ and $\text{int}(Z) \subset Z$ and so $\text{int}(Y) \cap \text{int}(Z)$ is an open subset of X contained in $Y \cap Z$. As $\text{int}(Y \cap Z)$ is the largest open subset of X contained in $Y \cap Z$, it follows that the RHS of (4) is contained in the LHS. This is (4). $\square$

There is a dual notion to the notion of interior.

Definition-Lemma 4.3. *Let $(X, \mathcal{T})$ be a topological space.*

*Let $Y \subset X$ be a subset of X . The **closure** of Y , denoted $\overline{Y}$, is the smallest closed subset A of X containing Y .*

Proof. The only thing we have to check is that there is a smallest closed subset of X containing Y . But this is clear, since the intersection of all subsets of X containing Y is a closed subset of X , containing Y , that contains all closed subsets of X , containing Y . $\square$

Here are some simple results about the closure:

Lemma 4.4. *Let $(X, \mathcal{T})$ be a topological space and let Y and Z be two subsets of X .*

- (1) *Y is closed if and only if $\overline{Y} = Y$.*
- (2) *$\overline{X} = X$ and $\overline{\emptyset} = \emptyset$.*
- (3) *If $Y \subset Z$ then $\overline{Y} \subset \overline{Z}$.*
- (4)

$$\overline{Y \cup Z} = \overline{Y} \cup \overline{Z}.$$

Proof. There are two ways to proceed (details left to the reader in both cases). The first is simply to mimic the proof of (4.2).

Aliter: Use (4.5) below. □

Lemma 4.5. *Let $(X, \mathcal{T})$ be a topological space and let Y be a subset.*

- (1)

$$\text{int}(X \setminus Y) = X \setminus \overline{Y}.$$

- (2)

$$X \setminus \text{int}(Y) = \overline{(X \setminus Y)}.$$

Proof. Let U be the LHS of (1). Then U is the biggest open subset of X contained in $X \setminus Y$. It follows that $A = X \setminus U$ is the smallest closed subset of X that contains Y . Thus $A = \overline{Y}$. Thus (1) holds.

(2) is similar. □

Definition 4.6. *Let $(X, \mathcal{T})$ be a topological space.*

*Let $Y \subset X$ be a subset of X . The **boundary** of Y , denoted ∂Y , is the complement of the interior of Y inside the closure of Y ,*

$$\partial Y = \overline{Y} \setminus \text{int}(Y).$$

Here are some simple results about the boundary:

Lemma 4.7. *Let $(X, \mathcal{T})$ be a topological space and let Y be a subset of X .*

- (1) *The closure of Y is a disjoint union of the interior of Y and the boundary of Y ,*

$$\overline{Y} = \text{int}(Y) \amalg \partial Y$$

- (2) *X is a disjoint union of the interior of Y , the boundary of Y and the interior of the complement of Y ,*

$$X = \text{int}(Y) \amalg \partial Y \amalg \text{int}(X \setminus Y).$$

- (3) *∂Y is closed.*
- (4) *$\partial Y = \partial(X \setminus Y)$.*

$$(5) \partial X = \partial \emptyset = \emptyset.$$

Proof. (1) is immediate from the definition and then (2) follows from (1) and (4.5).

By (2), ∂Y is the complement of the union of two open sets. This gives (3).

(2) applied to $X \setminus Y$ says

$$X = \text{int}(X \setminus Y) \amalg \partial(X \setminus Y) \amalg \text{int}(Y).$$

Comparing with the original formula gives (4).

As $\text{int}(X) = X$ it follows that $\partial X = \emptyset$. (4) applied to $\emptyset$ implies the rest of (5). $\square$

Example 4.8. *Let*

$$Y = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}.$$

Note that Y contains the open ball of radius one about the origin

$$B_1(0) = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \} \subset \mathbb{R}^2.$$

It follows that

$$B_1(0) \subset \text{int}(Y).$$

Now consider

$$V = \mathbb{R}^2 \setminus Y = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 1 \}.$$

We are going to show that V is open. Suppose that

$$p = (a, b) \in V.$$

Let

$$d(0, p) = r.$$

By assumption $r > 1$. Let

$$\rho = r - 1.$$

Let

$$U = B_\rho(p).$$

If $q \in U$ then

$$d(p, q) < \rho \quad \text{and} \quad d(0, p) \leq d(0, q) + d(q, p).$$

It follows that

$$\begin{aligned} d(0, q) &\geq d(0, p) - d(q, p) \\ &> r - \rho \\ &= 1. \end{aligned}$$

Thus $q \notin Y$. It follows that $q \in U \subset V$. But then V is open. It follows that Y is closed.

Let

$$C = S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}.$$

We check that C is the boundary of Y . We have to check that if $p = (x, y) \in C$ then p does not lie in the interior of Y . Suppose that $p \in U$ is open. We have to check that U is not contained in Y (or what comes to the same thing, that U intersects V). To this end, there is no harm in replacing U by a smaller open subset containing p . Thus we might as well assume that

$$U = B_r(p).$$

Let

$$p' = (1 + r/2)(x, y).$$

Then

$$d(0, p') = 1 + r/2 > 1 \quad \text{so that} \quad p' \notin Y.$$

On the other hand,

$$d(p, p') = \frac{r}{2} < r \quad \text{so that} \quad p' \in U.$$

Thus $p \notin \text{int}(Y)$. Thus

$$\text{int}(Y) = B_1(0), \quad \partial Y = S^1 \quad \text{and} \quad \text{int}(\mathbb{R}^2 - Y) = V.$$

Definition 4.9. Let $(X, \mathcal{T})$ be a topological space.

We say that $Y \subset X$ is **dense** in X if the closure of Y is the whole of X ,

$$\overline{Y} = X.$$

Lemma 4.10. Let $(X, \mathcal{T})$ be a topological space.

Then Y is dense if and only if every open subset U of X intersects Y (that is, the intersection $U \cap Y$ is non-empty).

Proof. (2) of (4.7) implies that Y is dense if and only if $\text{int}(X \setminus Y)$ is empty. This means that there are no open subsets of X contained in $X \setminus Y$. This is equivalent to saying every open subset intersects Y . $\square$

Definition 4.11. Let $(X, \mathcal{T})$ be a topological space.

We say that $Y \subset X$ is a **neighbourhood** of $x \in X$ if $x \in \text{int}(Y)$.

Lemma 4.12. Let $(X, \mathcal{T})$ be a topological space.

Then $Y \subset X$ is a neighbourhood of $x \in X$ if and only if there is an open subset U such that $x \in U \subset Y$.

Proof. An almost immediate consequence of the definitions. $\square$