

### 3. METRIC SPACES

There is another way to describe a topology.

**Definition 3.1.** Let  $(X, \mathcal{T})$  be a topological space.

A **basis**  $\mathcal{B}$  for  $\mathcal{T}$  is a collection of open subsets such that every element of  $\mathcal{T}$  is a union of elements of  $\mathcal{B}$ .

We are now ready to state our first non-trivial result:

**Lemma 3.2.** Let  $X$  be a set and let  $\mathcal{B} \subset \mathcal{P}(X)$ .

Then  $\mathcal{B}$  is a basis for a topology  $\mathcal{T}$  if and only if

- (1) If  $x \in X$  then we may find  $U \in \mathcal{B}$  such that  $x \in U$ .
- (2) If  $U_1$  and  $U_2 \in \mathcal{B}$  and  $x \in U_1 \cap U_2$  then there is  $U_3 \in \mathcal{B}$  such that

$$x \in U_3 \subset U_1 \cap U_2.$$

Before we give the proof it might help to introduce some useful notation.

**Definition 3.3.** Let  $X$  be a set. We say that a collection of subsets  $X_i$  **covers**  $X$  if  $X$  is the union,

$$X = \bigcup_i X_i.$$

**Lemma 3.4.** Let  $X$  be a set and let  $X_i$  be a collection of subsets.

The following are equivalent

- (1) The collection  $X_i$  covers  $X$ .
- (2) For every  $x \in X$  we may find an index  $i$  such that  $x \in X_i$ .

*Proof.* Obvious. □

Thus condition (1) above says that the elements of  $\mathcal{B}$  cover  $X$  and condition (2) says that the elements of  $\mathcal{B}$  contained in  $U_1 \cap U_2$  cover  $U_1 \cap U_2$ .

*Proof of (3.2).* Suppose that  $\mathcal{B}$  is a basis for the topology  $\mathcal{T}$ .

As  $X \in \mathcal{T}$  is open it follows that  $X$  is a union of elements of  $\mathcal{B}$ , that is, the elements of  $\mathcal{B}$  cover  $X$ . Thus (1) holds.

Now suppose that  $U_1$  and  $U_2 \in \mathcal{B}$ . Then  $U_1$  and  $U_2$  are open. It follows that  $U_1 \cap U_2$  is open. But then  $U_1 \cap U_2$  is the union of elements of  $\mathcal{B}$ . Thus (2) holds.

Now suppose that (1) and (2) hold. Let  $\mathcal{T}$  be the collection of all subsets of  $X$  which are the union of elements of  $\mathcal{B}$ . We show that  $\mathcal{T}$  is a topology.

$\emptyset$  is the union of no elements of  $\mathcal{B}$ . Thus  $\emptyset \in \mathcal{T}$ . The union of all elements of  $\mathcal{B}$  is the whole of  $X$ , by (1). Thus  $X \in \mathcal{T}$ .

Thus  $\mathcal{T}$  satisfies (1) of (2.3).

Now suppose that  $U_i, i \in I$ , is a collection of elements of  $\mathcal{T}$ . By assumption, we may find subsets  $B_{i\alpha} \in \mathcal{B}$  such that  $U_i$  is a union of  $B_{i\alpha}$ ,

$$U_i = \bigcup_{\alpha \in I_i} B_{i\alpha}.$$

In this case

$$\begin{aligned} U &= \bigcup_{i \in A} U_i \\ &= \bigcup_{i \in A} \bigcup_{\alpha \in I_i} B_{i\alpha} \\ &= \bigcup_{i \in A, \alpha \in I_i} B_{i\alpha} \in \mathcal{T}. \end{aligned}$$

Thus  $\mathcal{T}$  satisfies (2) of (2.3).

Now suppose that  $U$  and  $V \in \mathcal{T}$ . Then we may find  $U_\alpha$  and  $V_\beta \in \mathcal{B}$ , such that

$$U = \bigcup_{\alpha} U_\alpha \quad \text{and} \quad V = \bigcup_{\beta} V_\beta.$$

Then  $U_\alpha \cap V_\beta$  is a union of elements of  $\mathcal{B}$ , as we are assuming (2). Thus  $U_\alpha \cap V_\beta \in \mathcal{T}$ . But then

$$\begin{aligned} U \cap V &= \left( \bigcup_{\alpha} U_\alpha \right) \cap \left( \bigcup_{\beta} V_\beta \right) \\ &= \bigcup_{\alpha, \beta} (U_\alpha \cap V_\beta) \in \mathcal{T}, \end{aligned}$$

by what we already proved. Thus  $\mathcal{T}$  satisfies (3) of (2.3).

It follows that  $\mathcal{T}$  is a topology. □

We now come to the most important topological space of Math 190A:

**Definition-Lemma 3.5.** *Let*

$$\mathcal{B} = \{ (a, b) \subset \mathbb{R} \mid a < b, a, b \in \mathbb{R} \}$$

*the collection of all open intervals.*

*Then  $\mathcal{B}$  is a basis for a topology on  $\mathbb{R}$  called the **Euclidean topology**.*

*Proof.* We have to show that  $\mathcal{T}$  satisfies (1) and (2) of (3.2).

Suppose that  $r \in \mathbb{R}$ . Then  $r \in (r - 1/2, r + 1/2) \in \mathcal{B}$ . This is (1).

Suppose that  $U$  and  $V \in \mathcal{B}$ . Then  $U = (a, b)$  and  $V = (c, d)$ . Suppose that  $r \in (a, b) \cap (c, d)$ . Then  $a < r < b$  and  $c < r < d$ . Let

$$\alpha = \max(a, c) \quad \text{and} \quad \beta = \min(b, d).$$

Then

$$\alpha < r < \beta,$$

so that

$$r \in (\alpha, \beta) \in \mathcal{B}.$$

Thus (2) holds. In fact, since  $\alpha$  and  $\beta$  are independent of  $r$ , we have shown much more, that

$$(\alpha, \beta) = (a, b) \cap (c, d). \quad \square$$

We can generalise this example.

**Definition 3.6.** Let  $X$  be a set. A **metric**  $d$  on  $X$  is a function

$$d: X \times X \longrightarrow \mathbb{R}$$

such that

- (1)  $d(x, y) \geq 0$  for all  $x$  and  $y$ , with equality if and only if  $x = y$ .
- (2)  $d(x, y) = d(y, x)$  for all  $x$  and  $y$ .
- (3)

$$d(x, z) \leq d(x, y) + d(y, z)$$

for all  $x, y$  and  $z$ .

The pair  $(X, d)$  is called a **metric space**.

It is useful to think of a metric as the distance between two points. (1) then says that the distance between two points is non-negative and the distance is zero if and only if the two points are the same. (2) then says that the distance between  $x$  and  $y$  is the same as the distance between  $y$  and  $x$ . Finally (3) says that the distance between  $x$  and  $z$  is at most the distance between  $x$  and  $y$  and  $y$  and  $z$ . This is often referred to as the triangle inequality.

**Example 3.7.** Let  $X = \mathbb{R}$  and let  $d(x, y) = |x - y|$ .

Then  $(\mathbb{R}, d)$  is a metric space. Properties (1) and (2) are clear. We can check property (3) by hand or we can appeal to a more general result proved below.

**Definition 3.8.** Let  $(X, d)$  be a metric space. Let  $a \in X$  be a point of  $X$  and let  $r > 0$  be a real number. The **open ball** of radius  $r$  centred at  $a$  is the set

$$B_r(a) = \{x \in X \mid d(a, x) < r\}.$$

**Theorem 3.9.** *Let  $(X, d)$  be a metric space.*

*Then the set of all open balls*

$$\mathcal{B} = \{ B_r(a) \mid a \in X, r > 0 \}$$

*is the basis for a topology on  $X$ .*

*Proof.* Let  $a \in X$ . Then  $a \in B_1(a)$ . Thus (1) of (3.1) holds.

Now suppose that we have two open balls  $B_r(a)$ ,  $B_s(b)$  and that  $c \in B_r(a) \cap B_s(b)$ . Let  $\rho = d(a, c)$ . Then  $\rho < r$ , as  $c \in B_r(a)$ . Let  $\tau = r - \rho$ . Suppose that  $x \in B_\tau(c)$ . Then

$$\begin{aligned} d(a, x) &\leq d(a, c) + d(c, x) \\ &< \rho + \tau \\ &= \rho + (r - \rho) \\ &= r. \end{aligned}$$

Thus  $x \in B_r(a)$ . It follows that

$$B_\tau(c) \subset B_r(a).$$

By symmetry

$$B_{\tau'}(c) \subset B_s(b),$$

where  $\tau' = s - d(b, c) > 0$ . If  $t = \min(\tau, \tau')$  then

$$B_t(c) \subset B_r(a) \cap B_s(b).$$

This is (2) and (3.2) implies that  $\mathcal{B}$  is the basis for a topology on  $X$ .  $\square$

**Definition-Proposition 3.10.** *Let  $X = \mathbb{R}^n$  and define*

$$d: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

*by the rule*

$$d(x, y) = \sqrt{\sum (x_i - y_i)^2}.$$

*Then  $(\mathbb{R}^n, d)$  is a metric space.*

Note that the open ball  $B_r(a)$  of radius  $r$  about  $a$  is the what is inside the sphere of radius  $r$  with centre  $a$

To show that  $d$  is a metric, we make some simplifying observations. The key observation is that in fact we have a norm, the distance between two points is translation invariant.

Recall that the inner product of two vectors  $x$  and  $y$  in  $\mathbb{R}^n$  is

$$\langle x, y \rangle = \sum x_i y_i.$$

In these terms the distance of the vector  $x$  to the origin is simply

$$d(0, x) = \sqrt{\langle x, x \rangle}.$$

It is convenient to denote the RHS by the norm

$$||x|| = \sqrt{\langle x, x \rangle}.$$

We will need the following famous result:

**Lemma 3.11** (Cauchy-Schwarz-Bunjakowski). *Let  $x$  and  $y \in \mathbb{R}^n$ . Then*

$$|\langle x, y \rangle| \leq ||x|| ||y||$$

*with equality if and only if  $x$  and  $y$  are parallel.*

*Proof.* If  $y$  is the zero vector then both sides are zero and the inequality holds. In this case  $x$  and  $y$  are parallel.

Thus we may assume that  $y \neq 0$ . Let

$$\lambda = \frac{\langle x, y \rangle}{\langle y, y \rangle} \quad \text{and} \quad z = x - \lambda y.$$

Note that  $z$  is orthogonal to  $y$ ,

$$\begin{aligned} \langle z, y \rangle &= \langle x - \lambda y, y \rangle \\ &= \langle x, y \rangle - \lambda \langle y, y \rangle \\ &= 0. \end{aligned}$$

Thus Pythagoras applied to the triangle with sides

$$x, \quad \lambda y \quad \text{and} \quad z$$

gives

$$\begin{aligned} |\langle x, x \rangle| &= |\langle \lambda y, \lambda y \rangle| + |\langle z, z \rangle| \\ &= \lambda^2 |\langle y, y \rangle| + |\langle z, z \rangle| \\ &\geq \lambda^2 |\langle y, y \rangle|. \end{aligned}$$

It follows that

$$\langle x, y \rangle^2 \leq |\langle x, x \rangle| |\langle y, y \rangle|.$$

If we take the square root of both sides then we get the inequality.

Note that we have equality if and only if  $z = 0$ . If  $z = 0$  then  $x$  and  $y$  are certainly parallel. Now suppose that  $x$  and  $y$  are parallel. Then  $x = \mu y$  for some  $\mu$ . Taking the norm, it is easy to see that  $\mu = \lambda$ , so that  $z = 0$ .  $\square$

**Definition-Lemma 3.12.** *The function  $||x||$  on  $\mathbb{R}^n$  is a norm, that is, it is a function*

$$\mathbb{R}^n \longrightarrow \mathbb{R}$$

*such that*

- (1)  $||x|| \geq 0$  with equality if and only if  $x = 0$ .
- (2)  $||\lambda x|| = |\lambda| ||x||$ .

(3)

$$||x + y|| \leq ||x|| + ||y||$$

for all  $x$  and  $y$ .

*Proof.* (1) and (2) are immediate. We have

$$\begin{aligned} ||x + y||^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \\ &\leq \langle x, x \rangle + 2|\langle x, y \rangle| + \langle y, y \rangle \\ &\leq ||x||^2 + 2||x||^2||y||^2 + ||y||^2 \\ &= (||x|| + ||y||)^2. \end{aligned}$$

If we take square roots then we get (3). □

*Proof of (3.10).* Suppose that we are given three vectors  $x$ ,  $y$  and  $z \in \mathbb{R}^n$ . Let  $u = y - x$  and  $v = z - y$ . We have

$$\begin{aligned} d(x, z) &= ||z - x|| \\ &= ||u + v|| \\ &\leq ||u|| + ||v|| \\ &\leq d(x, y) + d(y, z). \end{aligned} \quad \square$$