

2. TOPOLOGICAL SPACES

To answer questions in topology, it becomes clear that we first need to introduce the appropriate language to talk about topological issues. The definition of a topology is surprisingly abstract and it takes quite a while to become comfortable with the definition. On the other hand, we will see later that the definition of continuity is surprisingly simple.

We recall a basic notion from set theory.

Definition 2.1. *Let X be a set. The **power set** of X , denoted $\mathcal{P}(X)$, is the set of all subsets of X .*

Example 2.2. *Suppose that*

$$X = \{a, b, c\}.$$

Then the power set of X , $\mathcal{P}(X)$, consists of all subsets of X . Note that a subset of X has zero, one, two or three elements. There is one subset with zero elements, the empty set $\emptyset$, there are three subsets with one element,

$$\{a\}, \quad \{b\} \quad \text{and} \quad \{c\},$$

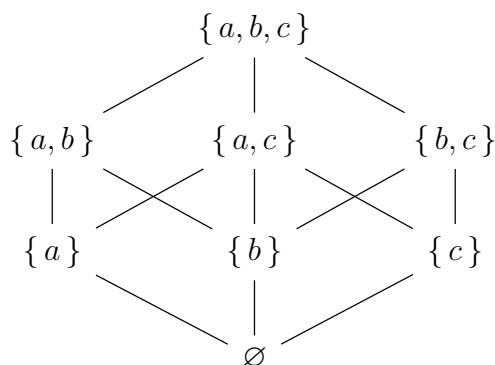
there are three subsets with two elements

$$\{b, c\}, \quad \{a, c\} \quad \text{and} \quad \{a, b\},$$

and there is one subset with three elements, X itself. Thus

$$\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}, \{a, b, c\}\}.$$

It is interesting to draw the lattice of all subsets.



In terms of the axioms of set theory, the power set axiom is one of the most interesting axioms, since it is the one axiom (other than the union) that produces bigger sets. Together with the axiom of infinity, the power set axiom implies the existence of infinite cardinals.

Definition 2.3. Let X be a set. A **topology** $\mathcal{T}$ on X is a collection of subsets of X , $\mathcal{T} \subset \wp(X)$, satisfying three axioms

- (1) $\emptyset$ and X belong to $\mathcal{T}$.
- (2) The union of any collection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.
- (3) The intersection of two elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

A **topological space** consists of any pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X .

Given a topology $\mathcal{T}$, we call the elements of $\mathcal{T}$ the **open** sets. Thus a topology is simply a collection of subsets of X , called the open sets, such that the empty set and the whole set X are open subsets, the arbitrary union of open subsets is open and the intersection of two open subsets is open.

Given any abstract definition, it might be useful to ask for the minimal example.

Example 2.4. Let $X = \emptyset$.

There is one topology on the empty set. There is one open subset, the empty set (note that the empty subset of the power set is not a topology, by the first axiom), so that $\mathcal{T} = \{\emptyset\}$. It is easy to check the axioms hold.

It is also interesting to ask for examples that always exist.

Example 2.5. Let X be a set.

Then $(X, \{\emptyset, X\})$ is a topological space. The only open subsets are the empty set and the whole set X . This is sometimes called the **trivial topology**. At the other extreme we have $(X, \wp(X))$. Every subset is open. This is sometimes called the **discrete topology**.

Definition 2.6. Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X . We say that $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$.

Vice-versa, we say that $\mathcal{T}_1$ is **finer** than $\mathcal{T}_2$ if $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$. Let $\mathcal{T}$ be any topology on X . Note that

$$\{\emptyset, X\} \quad \text{is always coarser than} \quad \mathcal{T}$$

and

$$\mathcal{T} \quad \text{is always coarser than} \quad \wp(X).$$

Let us now consider the possible topologies on a set with two elements:

Example 2.7. Let $X = \{a, b\}$.

We consider the open subsets with one element. If there are none then we have the smallest possible topology, the trivial topology:

$$\mathcal{T} = \{\{a, b\}, \emptyset\}.$$

If there are two then we have the biggest possible topology, the discrete topology:

$$\mathcal{T} = \{\{a, b\}, \{a\}, \{b\}, \emptyset\}.$$

Suppose that there is one. We may suppose that $\{a\}$ is open. In this case we have

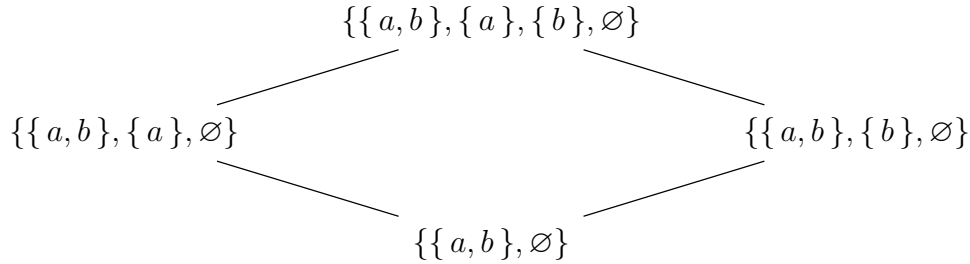
$$\mathcal{T} = \{\{a, b\}, \{a\}, \emptyset\}.$$

It is not hard to see that this satisfies the axioms. By symmetry we also have

$$\mathcal{T} = \{\{a, b\}, \{b\}, \emptyset\}.$$

This exhausts all possible topologies on a two element set.

It is interesting to draw the lattice of all topologies:



It is convenient to specify topologies in different ways.

Definition 2.8. Let $(X, \mathcal{T})$ be a topological space.

We say that a subset A of X is **closed** if the complement $U = X \setminus A$ is open.

Clearly we can specify a topological space by giving the closed subsets. The closed subsets have the following properties:

- (1) $\emptyset$ and X are closed subsets.
- (2) The arbitrary intersection of closed subsets is closed.
- (3) The union of two closed subsets is closed.

This an immediate consequence of

Lemma 2.9. Let X be a set.

- (1) $X \setminus \emptyset = X$ and $X \setminus X = \emptyset$.
- (2)

$$X \setminus \left(\bigcup_{\alpha} U_{\alpha} \right) = \bigcap_{\alpha} (X \setminus U_{\alpha}).$$

(3)

$$X \setminus \left(\bigcap_{\alpha} U_{\alpha} \right) = \bigcup_{\alpha} (X \setminus U_{\alpha}).$$

Example 2.10. Let X be a set.

Define a topology on X by declaring that the finite sets and X itself are the closed subsets. Note that the arbitrary intersection of finite sets is finite. Note also that the union of two finite sets is finite. In this topology the open sets are the complements of finite sets and the empty set.

We can generalise this example a little bit. To this end, recall some definitions and results from set theory.

Definition 2.11. Let X and Y be two sets. We say that X and Y have the same **cardinality** if there is a bijection

$$f: X \longrightarrow Y.$$

We say that $\aleph$ is a **cardinal** if it is an equivalence class of sets of the same cardinality (this is not quite right. The collection of all sets of the same cardinality is a class, not a set [this is an instance of Russell's paradox]. However it is not too hard to fix this problem. Recall that sets have a rank, an ordinal. Let α be the smallest ordinal such that there is a set of the given cardinality of rank α . Then $\aleph$ consists of all sets of the same cardinality of rank α . Then $\aleph$ is a set).

Thus X has cardinality $\aleph$ if X is in bijection with some element of $\aleph$.

Definition 2.12. If $\aleph$ and $\beth$ are two cardinals then we write $\aleph \leq \beth$ if there are two sets X and Y of cardinality $\aleph$ and $\beth$ and there is an injective function $f: X \longrightarrow Y$.

Theorem 2.13 (Schröder–Bernstein). Let $\aleph$ and $\beth$ be two cardinals.

If $\aleph \leq \beth$ and $\beth \leq \aleph$ then $\aleph = \beth$.

Proof. By assumption there are sets X and Y of cardinality $\aleph$ and $\beth$ and injective functions $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$. It follows by the Schröder–Bernstein theorem that there is a bijection $h: X \longrightarrow Y$. In this case $\aleph = \beth$. $\square$

Now assume the axiom of choice. More generally, if we fix an infinite cardinal $\aleph$, then we can form a topology on X by declaring the closed subset to be the subsets of cardinality at most $\aleph$, plus X itself.

It is clear that the intersection of arbitrary subsets of cardinality at most $\aleph$ has cardinality at most $\aleph$. On the other hand, the union of two subsets of cardinality $\aleph$ has cardinality $\aleph$ (this is false unless we

assume the axiom of choice. In fact without the axiom of choice it is possible that there are infinite sets A such that if $B \subset A$ then either B is finite or $A \setminus B$ is finite. These are called amorphous).

Note that the open sets are the empty set and all sets whose complement has cardinality at most $\aleph$.

We can vary this example just a little bit. Instead of taking all sets of cardinality at most $\aleph$ as closed, we could take all sets of cardinality strictly less than $\aleph$. In the case we take $\aleph = \aleph_0$, the cardinality of $\mathbb{N}$, the smallest infinite cardinal, then we get back the example above where the closed sets (apart from X) are the finite sets.