

19. MANIFOLDS

Definition 19.1. A *topological manifold* of dimension n is a Hausdorff topological space X that is second countable and every point has an open neighbourhood homeomorphic to $\mathbb{R}^n$.

In fact the definition of a topological manifold has slight variations in the literature.

Definition 19.2. A topological space is *second countable* if it has a countable basis.

Example 19.3. $\mathbb{R}^n$ is an n -manifold.

$\mathbb{R}^n$ is Hausdorff and it is certainly homeomorphic to $\mathbb{R}^n$. Note that

$$\mathcal{B} = \{ B_r(a) \mid r \in \mathbb{Q}, a \in \mathbb{Q}^n \}$$

is a countable basis. Indeed given any open ball with irrational centre and irrational radius, it is easy to find an open ball with rational centre and rational radius that contains the irrational centre. Thus $\mathcal{B}$ is a basis. On the other hand, $\mathcal{B}$ is in bijection with $\mathbb{Q}^{n+1}$ and so it is certainly countable.

More generally any open subset U of $\mathbb{R}^n$ is a topological n -manifold (this is a place where one variation arises, sometimes a topological manifold is required to be connected). Indeed, every point has a neighbourhood homeomorphic to an open ball. Inside that open ball is an open cube. But all open cubes are homeomorphic to $\mathbb{R}^n$ (indeed all open balls are homeomorphic to $\mathbb{R}^n$).

The circle, S^1 is a curve (a connected 1-manifold), the torus, $S^1 \times S^1$, the Klein bottle, and the real projective plane $\mathbb{RP}^2$ are surfaces (or connected 2-manifolds). It is not hard to check that all four spaces are locally isomorphic to $\mathbb{R}^1$ or $\mathbb{R}^2$. We just have to check what happens on the boundary. But locally we simply identify the union of two half open balls, union the boundary diameter. These diameters are identified and we get something homeomorphic to an open ball. So every point has an open neighbourhood homeomorphic to an open ball and we already saw this means every point has an open neighbourhood homeomorphic to $\mathbb{R}^n$.

The open neighbourhoods homeomorphic to an open ball form an open cover of our topological spaces. All of these spaces are compact. Therefore there is a finite cover. But then it is clear that these spaces are second countable.

To check that the Hausdorff condition holds takes a little bit more work. We have to check what happens in a neighbourhood of any point

where there is identification in every neighbourhood of that point. This is not too hard.

We also saw in the homework that S^n is an n -manifold. Indeed, if you remove the North pole then what is left is homeomorphic to $\mathbb{R}^n$, by stereographic projection.

Theorem 19.4. *A curve X is homeomorphic to either $\mathbb{R}$ or S^1 .*

Note that these two spaces are not homeomorphic, since $\mathbb{R}$ is not compact and S^1 is compact.

We have already seen an example that shows that the Hausdorff condition can fail and yet the other conditions hold. If we take the example of §18, where we identify two copies of the real line, except over the origin, then the space we get is covered by two open sets homeomorphic to $\mathbb{R}$. It is therefore certainly second countable.

It is a little harder to give an example where the second countable assumption is needed.

Example 19.5. *Recall that ω_1 is the first uncountable ordinal.*

Consider the product

$$\omega_1 \times [0, 1).$$

Order this set lexicographically:

$$(\alpha, r) < (\beta, s) \quad \text{if and only if} \quad \text{either } \alpha < \beta \text{ or } \alpha = \beta \text{ and } r < s.$$

Let X be the set you get by deleting the smallest element, namely $(0, 0)$. As X is an ordered set, it follows that X inherits a topology.

Lemma 19.6. *X is a path-connected Hausdorff topological space that is locally homeomorphic to $\mathbb{R}$ but X is not a curve.*

Proof. First some notation. If we are given two totally ordered sets A and B we say that A and B have the same **order type** if there is a bijection

$$f: A \longrightarrow B$$

from A to B that preserves the order, that is,

$$\text{if } a_1 \leq a_2 \quad \text{then} \quad f(a_1) \leq f(a_2).$$

The key is to prove that if we make the same construction for any ordinal smaller α smaller than ω_1 (and we don't throw away the smallest point) then we get an ordered set L_α with the same order type as $[0, 1)$.

We use transfinite induction. When $\alpha = 0$ then we get $L_0 = [0, 1)$. This is the start of the induction. Now suppose that $\alpha = \beta + 1$. Suppose

that L_β has the order type of $A = [0, 1)$. Then $L_\alpha = L_{\beta+1}$ has the order type of $B = [0, 1) \cup [1, 2)$. Note that

$$A \longrightarrow B \quad \text{given by} \quad x \longrightarrow 2x$$

is an order preserving bijection. Thus L_α has the order type of $[0, 1)$.

Now consider the case when α is a limit ordinal. Note that we can find ordinals

$$\alpha_1 < \alpha_2 < \alpha_3 < \dots$$

whose limit is α (this is true for any limit ordinal less than ω_1 , since every ordinal less than ω_1 is countable). We may assume that L_{α_n} has the same order type as $[0, n + 1)$. Thus

$$\begin{aligned} L_\alpha &= \bigcup L_{\alpha_n} \\ &= \bigcup [0, n + 1) \\ &= [0, \infty) \\ &= [0, 1). \end{aligned}$$

The first line to the second line and the third line to the fourth line are just equalities up to order types.

It follows by transfinite induction that L_α has the same order type as $[0, 1)$, for all $\alpha < \omega_1$. Thus if we delete the smallest point then we get a space X_α homeomorphic to $\mathbb{R}$. Note that X_α is open in X .

Suppose that $y = (\beta, s) \in X$. Pick $\beta < \alpha$. Then $y \in X_\alpha$. Thus y has a neighbourhood isomorphic to $\mathbb{R}$. Similarly if x and $y \in X$ then we may find α such that x and $y \in X_\alpha$. Thus X is Hausdorff and path-connected.

Note that the points

$$x_\alpha = (\alpha, \frac{1}{2})$$

have disjoint open neighbourhoods. For example

$$\{\alpha\} \times (0, 1)$$

is a neighbourhood of x_α that does not contain x_β , if $\beta \neq \alpha$. Thus any basis $\mathcal{B}$ must contain an open set containing x_α , not containing x_β , for all $\beta \neq \alpha$. As ω_1 is uncountable it follows that $\mathcal{B}$ is uncountable. Thus X is not second countable. $\square$

X is sometimes called the long line. X is like $(0, \infty)$, except that it extends further to the right. There are some interesting variations on this theme. One could include the smallest point. In this case we get a X_0 space that looks like $[0, \infty)$, except that it extends further to the right. One can also tack on a biggest point at the end.

One can create a line that goes infinitely in both directions. Take X , order it the other way and include a point in the middle. Or take two copies of X_0 and identify the two origins. We get a line X_1 that looks like $(-\infty, \infty)$, except that it extends further both to the right and to the left. Then one can tack on points at either end. These are all different (not even homeomorphic) and unfortunately sometimes each one is called the long line. Note that both X and X_1 are not curves only because they are not second countable.

It also interesting to consider what happens if we go one step further and do the same construction for $\omega_1 + 1$. This is not locally homeomorphic to $\mathbb{R}$, at the point $(\omega_1, 0)$.

The classification of surfaces is a little bit more involved.

Theorem 19.7. *Every two dimensional compact connected orientable manifold is homeomorphic to a sphere with g handles attached.*

A surface is orientable if one can consistently and continuously distinguish between left and right on the surface. The sphere and torus are orientable. The Klein bottle and Möbius strip are not. g is called the **genus**.

Given a surface S we can attach a **handle** as follows. The handle itself is simply a cylinder. Pick two circles on the surface (let's say in two disjoint open neighbourhoods homeomorphic to $\mathbb{R}^2$). We orient the boundary circles of the cylinder and the two circles consistently with the orientation on the surfaces (S or the cylinder). Now attach the circles (more formally take the quotient of an appropriate equivalence relation). The resulting topological space is a connected, compact, orientable, two dimensional manifold and its genus is one more than the genus of S .

Unoriented surfaces are closely related to oriented surfaces (in fact every unoriented surface has a double cover that is oriented).