

16. A LITTLE CATEGORY THEORY

Definition 16.1. A category $\mathcal{C}$ consists of two things. The first is the **objects** of the category. The second is, given any two objects, X and Y , we associate a collection of **morphisms**, denoted $\text{Hom}(X, Y)$. If the collection of all morphisms from X to Y is a set, we say that the category is **locally small**; it is **small** if in addition the objects are a set. Given three objects, X , Y and Z and two morphisms $f \in \text{Hom}(X, Y)$ and $g \in \text{Hom}(Y, Z)$, there is a rule of composition,

$$g \circ f \in \text{Hom}(X, Z).$$

A category satisfies the following two axioms.

- (1) Composition is associative. That is, given $f \in \text{Hom}(X, Y)$, $g \in \text{Hom}(Y, Z)$, $h \in \text{Hom}(Z, W)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (2) There is an element $I_X \in \text{Hom}(X, X)$ that acts as an identity, so that if $f \in \text{Hom}(X, Y)$ and $g \in \text{Hom}(Y, X)$ we have,

$$f \circ I_X = f,$$

and

$$I_X \circ g = g.$$

Given $f \in \text{Hom}(X, Y)$ we say that $g \in \text{Hom}(Y, X)$ is the **inverse** of f if $f \circ g = I_Y$ and $g \circ f = I_X$. In this case we say that f is an **isomorphism** and we say that X and Y are **isomorphic**.

Given most advanced mathematics classes, there is normally a naturally associated category.

For us, the most natural category is the category of topological spaces. The objects are topological spaces and the morphisms are continuous maps. Composition of morphisms is composition of functions. It is not hard to see that all the axioms are satisfied. Two topological spaces are then homeomorphic if and only if they are isomorphic as objects of this category. The category of metric spaces is a subcategory of the category of topological spaces.

Another very natural category is the category of sets. Here a morphism is any function. Two sets are isomorphic if and only if they have the same cardinality.

The category of vector spaces, is the category whose objects are vector spaces and whose morphisms are linear maps.

Yet another category is the category of groups. The objects are the collection of all groups and the morphisms are homomorphisms. Two

groups are isomorphic if and only if they are isomorphic as objects of the category.

Definition 16.2. *Let*

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ b \downarrow & & \downarrow c \\ C & \xrightarrow{d} & D \end{array}$$

be a collection of objects and morphisms belonging to a category.

*We say that the diagram **commutes** if the two morphisms from A to D are the same, that is*

$$c \circ a = d \circ b.$$

In a category, the focus of interest is not the objects, but the morphisms between the objects. We want to characterise isomorphism classes of objects, not in terms of the elements they contain but in terms of how they relate to all other objects.

Definition-Lemma 16.3. *Let $\mathcal{C}$ be a category and let X and Y be two objects of $\mathcal{C}$. The **categorical product** of X and Y , denoted $X \times Y$, is an object together with two morphisms $p: X \times Y \rightarrow X$ and $q: X \times Y \rightarrow Y$ that are universal amongst all such morphisms, in the following sense:*

Suppose that there are morphisms $f: Z \rightarrow X$ and $g: Z \rightarrow Y$. Then there is a unique morphism $h: Z \rightarrow X \times Y$ which makes the following diagram commute,

$$\begin{array}{ccccc} & & X & & \\ & f \nearrow & \uparrow p & & \\ Z & \xrightarrow{h} & X \times Y & & \\ & g \searrow & \downarrow q & & \\ & & Y & & \end{array}$$

The product of two topological spaces, with the product topology, is a categorical product. The categorical product is unique, up to a unique isomorphism.

Proof. We already showed that the product of two topological spaces, with the product topology, is a categorical product in (9.6). The only thing that is missing there, is that if we are given f and g , there is a unique function

$$h: Z \rightarrow X \times Y.$$

But this is in fact clear.

What is new is the idea of using commuting diagrams to characterise the product, up to isomorphism.

So suppose we are given two topological spaces P_1 and P_2 , both of which satisfy the universal property of a categorical product. In particular, it follows that we are given two continuous maps,

$$p_1: P_1 \longrightarrow X \quad \text{and} \quad q_1: P_1 \longrightarrow Y.$$

We feed this information into the universal property of P_2 . It follows that there is a continuous map

$$h_{12}: P_1 \longrightarrow P_2$$

which maps the following diagram commute:

$$\begin{array}{ccc} & & X \\ & \nearrow p_1 & \uparrow p_2 \\ P_1 & \overset{h_{12}}{\dashrightarrow} & P_2 \\ & \searrow q_1 & \downarrow q_2 \\ & & Y \end{array}$$

Now run the same argument with the roles of P_1 and P_2 switched. This produces a continuous map

$$h_{21}: P_2 \longrightarrow P_1$$

which maps another big diagram commute. Consider the composition

$$h_{21} \circ h_{12}: P_1 \longrightarrow P_1.$$

Putting together the two commuting diagrams, we get another commuting diagram, but in this commuting diagram P_2 makes no appearance.

$$\begin{array}{ccc} & & X \\ & \nearrow p_1 & \uparrow p_1 \\ P_1 & \overset{\quad}{\dashrightarrow} & P_1 \\ & \searrow q_1 & \downarrow q_1 \\ & & Y \end{array}$$

Note that one function that makes this diagram commute is the identity

$$i_{P_1}: P_1 \longrightarrow P_1.$$

Another is $h_{21} \circ h_{12}$. By the universal property of P_1 there is only one function that does the trick. By uniqueness, we therefore have

$$h_{21} \circ h_{12} = i_{P_1}.$$

By symmetry

$$h_{12} \circ h_{21} = i_{P_2}.$$

Thus h_{12} is a homeomorphism. In particular P_1 and P_2 are homeomorphic. But in fact, we did not need to make any choices in our construction of h_{12} . Thus P_1 and P_2 are homeomorphic, up to a unique homeomorphism. $\square$