

15. COMPACT AND HAUSDORFF

Lemma 15.1. *Let X be a compact set.*

If $F \subset X$ is closed then F is compact.

Proof. Let $U = X \setminus F$. Then U is open in X . Let

$$\mathcal{V} = \{V_\alpha \mid \alpha \in \Lambda\}$$

be an open cover of Y . Then we may find U_α open in X such that

$$V_\alpha = U_\alpha \cap X.$$

Let

$$\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\} \cup \{U\}.$$

It is easy to see that $\mathcal{U}$ is an open cover of X . As X is compact, this open cover has a finite subcover.

$$\mathcal{U}_0 = \{U_\beta \mid \beta \in M\} \cup \{U\}.$$

(note that it is clear that U is never redundant). But then

$$\mathcal{V}_0 = \{V_\beta \mid \beta \in M\}$$

is an open cover of F . This is a finite subcover of the original cover $\mathcal{U}$. Thus F is compact. $\square$

Remark 15.2. *Strictly speaking the proof we have given contains a small gap.*

It is the same issue that appeared at the start of §5 in the proof that the subspace topology is a topology. The same fix there works here.

Aliter: We can also define $\mathcal{U}$ to be the set of all open sets of X such that if we intersect this set F then we get an element of $\mathcal{V}$. This again avoids an invocation of the axiom of choice.

Similar issues appear in (15.4) and (14.9).

We recall a definition introduced in Homework 3:

Definition 15.3. *Let $(X, \mathcal{T})$ be a topological space. We say that $(X, \mathcal{T})$ is **Hausdorff** if for any two points $x \neq y \in X$ we may find two disjoint open sets U and V .*

Lemma 15.4. *Let X be a Hausdorff topological space and let Y be a subspace.*

If Y is compact then Y is closed in X .

Proof. Let $x \in X \setminus Y$ be a point of X not in Y and let $y \in Y$ be a point of Y . By assumption we may find $x \in U_y$ and $y \in V_y$, where U_y and V_y are open and disjoint. As we vary $y \in Y$ we get an open cover of Y ,

$$\mathcal{V} = \{V_y \cap Y \mid y \in Y\}.$$

As Y is compact this open cover has a finite subcover

$$\mathcal{W} = \{ V_{y_i} \cap Y \mid 1 \leq i \leq k \}.$$

Let

$$U = \bigcap_{i=1}^k U_{y_i} \quad \text{and} \quad V = \bigcup_{i=1}^k V_{y_i}.$$

Then $x \in U$ and $Y \subset V$ are disjoint. V is open as the arbitrary union of open sets is open. U is open as it is the finite intersection of open sets.

In particular every point of $X \setminus Y$ has an open neighbourhood U . But then $X \setminus Y$ is open, so that Y is closed. $\square$

Definition 15.5. We say that $Y \subset \mathbb{R}^n$ is **bounded** if there is a real $r > 0$ such that

$$Y \subset B_r(0).$$

In words, Y is bounded if it is contained in some open ball centred about the origin.

More generally, it makes sense to talk about bounded subsets of any metric space (X, d) . In this case one picks a reference point (like the origin) and then one shows the definition does not depend on the reference point.

Theorem 15.6 (Heine-Borel). *A subspace Y of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Suppose that Y is compact. $\mathbb{R}^n$ is a metric space and so it is Hausdorff. It follows that Y is closed.

$$\{ B_n(0) \cap Y \mid n \in \mathbb{N} \}$$

is an open cover of Y , as

$$\{ B_n(0) \mid n \in \mathbb{N} \}$$

is an open cover of $\mathbb{R}^n$. As Y is compact it has a finite subcover. But then Y is bounded.

Now suppose that Y is closed and bounded. Then

$$Y \subset B_0(r) \subset [-r, r]^n.$$

$[-r, r]$ is compact by (14.6) and so $[-r, r]^n$ is compact by (14.7) and induction on n . As Y is closed it is a closed subset of $[-r, r]^n$. But then Y is compact. $\square$

Example 15.7. Consider the following three subspaces of $\mathbb{R}$:

$$\mathbb{Z}, \quad \mathbb{Q} \cap [0, 1] \quad \text{and} \quad C.$$

$\mathbb{Z}$ is closed but not bounded and so it is not compact. $\mathbb{Q} \cap [0, 1]$ is bounded but not closed and so it is not compact. The Cantor set C is both closed and bounded and so it is compact.

Definition-Lemma 15.8. *Let (X, d) be a metric space.*

*If $A \subset X$ the **distance** of A to a point $x \in X$ is*

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

The function

$$f: X \longrightarrow \mathbb{R} \quad \text{given by} \quad f(x) = d(x, A)$$

is continuous. If A is compact then

$$f(x) = 0 \quad \text{if and only if} \quad x \in A.$$

Proof. Let $\epsilon > 0$. Let $\delta = \epsilon$. Suppose that x and y are two points of X such that

$$d(x, y) < \delta = \epsilon.$$

Let

$$r = \delta - d(x, y) > 0.$$

Pick $a \in A$ such that

$$d(x, a) < d(x, A) + r.$$

Then

$$\begin{aligned} f(y) &= d(y, A) \\ &\leq d(y, a) \\ &\leq d(x, y) + d(x, a) \\ &< d(x, A) + r + d(x, y) \\ &= f(x) + \epsilon. \end{aligned}$$

By symmetry, it follows that

$$|f(x) - f(y)| < \epsilon,$$

so that f is continuous.

Now suppose that A is compact. If $x \in A$ then

$$\begin{aligned} 0 &\leq f(x) \\ &= d(x, A) \\ &\leq d(x, x) \\ &= 0. \end{aligned}$$

Thus $f(x) = 0$.

Now suppose $x \notin A$. As A is compact the function

$$A \longrightarrow \mathbb{R} \quad \text{given by} \quad a \longrightarrow d(a, x)$$

achieves its minimum and maximum. We already saw the minimum is not zero. Thus $f(x) > 0$. $\square$

Definition-Theorem 15.9. *Let (X, d) be a metric space and let*

$$\mathcal{U} = \{ U_\alpha \mid \alpha \in \Lambda \}$$

*be an open cover of X . We say that $\delta > 0$ is a **Lebesgue number** for $\mathcal{U}$ if every open ball $B_\delta(a)$ of radius δ we may find an element U_α of the cover that contains this open ball*

$$B_\delta(a) \subset U_\alpha.$$

If X is compact then every open cover has a Lebesgue number.

Proof. As X is compact, we may assume that the cover is finite.

Note that

$$B_\delta(a) \subset U_\alpha \quad \text{if and only if} \quad d(a, X \setminus U_\alpha) \geq \delta.$$

Let

$$f_\alpha: X \longrightarrow \mathbb{R}$$

be the function

$$f_\alpha(x) = d(x, X \setminus U_\alpha).$$

Then f_α is continuous. As $X \setminus U_\alpha$ is closed and X is compact, it follows that $X \setminus U_\alpha$ is compact.

Define a function

$$f: X \longrightarrow \mathbb{R}$$

by the rule

$$f(x) = \max\{ f_\alpha \mid \alpha \in \Lambda \}.$$

Note that we really do have a maximum (and not just a supremum) as Λ is finite. Note that f is continuous as it is the maximum of finitely many continuous functions (exercise for the reader).

As f is continuous and X is compact, f achieves its maximum and minimum. Let δ be the minimum. Given $a \in X$ note that $a \in U_\alpha$, for some α , so that $f_\alpha(a) > 0$. But then $\delta > 0$. This is a Lebesgue number. $\square$