

14. COMPACTNESS

Recall that in (3.3) we said that a collection of subsets of a set X covers X if their union is the whole of X .

Definition 14.1. *Let*

$$\{X_\alpha \mid \alpha \in \Lambda\}$$

be a cover of X , so that

$$X = \bigcup_{\alpha \in \Lambda} X_\alpha.$$

We say that

$$\{X_\beta \mid \beta \in M\} \subset \{X_\alpha \mid \alpha \in \Lambda\}$$

*is a **subcover** if it is also a cover, so that*

$$X = \bigcup_{\beta \in M} X_\beta.$$

Note that $M \subset \Lambda$. One might think of a subcover this way. We are given a collection of sets that covers X . But perhaps there is some redundancy, perhaps we can still cover X with fewer subsets.

Example 14.2. *Let*

$$X = \{a, b, c\}.$$

Then

$$\{\{a\}, \{a, b\}, \{b, c\}\}$$

is a cover of X . Every element of X belongs to one of these three subsets.

But there is some obvious redundancy here. We could get away with less subsets:

$$\{\{a\}, \{b, c\}\}$$

is a subcover but so also is

$$\{\{a, b\}, \{b, c\}\}.$$

Note that we can not thin out the original collection anymore. We will need to use at least two sets and

$$\{\{a\}, \{a, b\}\}$$

is not a cover because c does not belong to these two subsets.

Note also that

$$\{\{a, c\}, \{a, b\}\}$$

is not a subcover, because $\{a, c\}$ is not one of the original three subsets.

We are now ready to state one of the most useful properties that any topological space might have:

Definition 14.3. Let X be a topological space.

An **open cover** of X is a cover by open subsets

$$\{U_\alpha \mid \alpha \in \Lambda\}$$

so that U_α is open and

$$X = \bigcup_{\alpha \in \Lambda} U_\alpha.$$

We say that X is **compact** if every open cover

$$\{U_\alpha \mid \alpha \in \Lambda\}$$

has a finite subcover

$$\{U_\beta \mid \beta \in M\}.$$

Note that finite subcover simply means that

$$\{U_\beta \mid \beta \in M\},$$

is finite. Equivalently the indexing set M of the subcover is finite.

Put differently, X is compact if every time you write down an infinite open cover, this cover has a huge amount of redundancy, we can get away with a subcover with finitely many elements.

Example 14.4. Let X be a finite topological space.

(Finite simply means the underlying set X is finite).

Then X is compact. Indeed any cover is a subset of the power set $\mathcal{P}(X)$ and this has finitely many elements.

We can push this example just a little bit further:

Example 14.5. Let $(X, \mathcal{T})$ be a topological space such that $\mathcal{T}$ is finite.

Then X is compact. Indeed any open cover is a subset of $\mathcal{T}$ and this has finitely many elements.

It is interesting to consider which intervals are compact:

Theorem 14.6. Let $Y \subset \mathbb{R}$ be an interval or half ray.

Then Y is compact if and only if Y is a closed interval.

Proof. Up to homeomorphism we have to consider three cases

$$(-\infty, \infty), \quad [0, \infty) \quad \text{and} \quad [0, 1].$$

We first show that the first two spaces are not compact. Let

$$\{(-n, n) \mid n \in \mathbb{N}\}.$$

This is an open cover of $(-\infty, \infty)$. This cover has infinitely many elements (it is clearly countable) and any subcover has infinitely many elements. In fact the elements of the cover are nested

$$(-1, 1) \subset (-2, 2) \subset (-3, 3) \subset \dots (-n, n) \dots$$

If we sampled only finitely many of these intervals the union would have the form

$$(-m, m)$$

for some m . This does not cover $(-\infty, \infty)$. Thus $(-\infty, \infty)$ is not compact.

Similarly,

$$\{ [0, n) \mid n \in \mathbb{N} \}.$$

is an open cover of $[0, \infty)$ with no finite subcover. Thus $[0, \infty)$ is not compact.

We now turn to showing that $[0, 1]$ is compact. Let

$$\mathcal{U} = \{ U_\alpha \mid \alpha \in \Lambda \}$$

be an open cover of $[0, 1]$. Let

$$\mathcal{U}^i = \{ I \mid I \subset U_\alpha, \alpha \in \Lambda \}$$

where I is an interval that is open in $[0, 1]$ (note that our definition of $\mathcal{U}^i$ avoids using the axiom of choice). Then

$$\mathcal{U}^i$$

is an open cover of $[0, 1]$ and if

$$\mathcal{U}^i$$

has a finite subcover then so does

$$\mathcal{U}.$$

Replacing

$$\mathcal{U} \quad \text{by} \quad \mathcal{U}^i,$$

we may therefore assume that our open cover is a cover by intervals.

Let

$$Y = \{ a \in [0, 1] \mid [0, a] \text{ is a subset of the union of finitely many elements of } \mathcal{U} \}.$$

Suppose that $a \in Y \cap U_\alpha$. By assumption $[0, a]$ is a subset of the union of finitely many elements of $\mathcal{U}$,

$$[0, a] \subset \bigcup_{i=1}^n V_i \quad \text{where} \quad V_i \in \mathcal{U}.$$

Consider the collection

$$\{ V_i \mid 1 \leq i \leq n+1 \}$$

where $V_{n+1} = U_\alpha$. Let $b \in U_\alpha$. As a and $b \in U_\alpha$ and U_α is an interval, it follows that the whole closed interval between a and b belongs to U_α . Thus

$$[0, b] = \bigcup_{i=1}^{n+1} V_i \quad \text{so that} \quad b \in Y.$$

But then

$$U_\alpha \subset Y.$$

We have shown that for every $\alpha \in \Lambda$, either

$$U_\alpha \subset Y \quad \text{or} \quad U_\alpha \subset [0, 1] \setminus Y.$$

It follows that both Y and $[0, 1] \setminus Y$ are open. As $[0, 1]$ is connected, it follows that either Y is the empty set or the whole of $[0, 1]$.

Pick α such that $0 \in U_\alpha$. It follows that $0 \in Y$. Thus Y is not empty and so $Y = [0, 1]$. Hence $[0, 1]$ is compact. $\square$

The argument above is somewhat subtle. The reader is invited to consider what happens for examples such as

$$[0, 1)$$

and the open cover

$$\left\{ \left[0, 1 - \frac{1}{n}\right) \mid n \in \mathbb{N} \right\}.$$

Theorem 14.7. *Let X and Y be two topological spaces.*

Then $X \times Y$ is compact if and only if X and Y are compact.

The proof will use two:

Lemma 14.8. *Let $f: X \longrightarrow Y$ be a continuous and surjective map of topological spaces.*

If X is compact then Y is compact.

Proof. Let

$$\mathcal{V} = \{ V_\alpha \mid \alpha \in \Lambda \}$$

be an open cover of Y . Then we get an open cover

$$\mathcal{U} = \{ U_\alpha = f^{-1}(V_\alpha) \mid \alpha \in \Lambda \}$$

of X . This has a finite subcover,

$$\{ U_\beta \mid \beta \in M \}.$$

As f is surjective we get a finite subcover of Y ,

$$\{ V_\beta \mid \beta \in M \}.$$

Thus Y is compact. $\square$

Lemma 14.9 (Tube Lemma). *Let X and Y be topological spaces. Fix a point $x \in X$ and let*

$$\mathcal{W} = \{ W_\alpha \mid \alpha \in \Lambda \}$$

be a collection of open sets whose union contains $\{x\} \times Y$.

If Y is compact then we may find a finite subcollection

$$\mathcal{W}_0 = \{ W_\beta \mid \beta \in M \}$$

and an open neighbourhood U_x of $x \in X$ such that the union of the elements of $\mathcal{W}_0$ contains $U_x \times Y$.

Proof. By definition of the product topology, for each $y \in Y$, we may find

$$(x, y) \in U_y \times V_y \subset W_\alpha$$

for some α , where U_y and V_y are open in X and Y . Thus

$$\mathcal{V} = \{ V_y \mid y \in Y \}$$

is an open cover of Y . As Y is compact there is a finite subcover

$$\mathcal{V}_0 = \{ V_{y_i} \mid 1 \leq i \leq k \}.$$

Let

$$U_x = \bigcap_{i=1}^k U_{y_i}.$$

By assumption, for each i , we may find α_i such that

$$U_{y_i} \times V_{y_i} \subset W_{\alpha_i}.$$

In this case

$$\mathcal{W}_0 = \{ W_{\alpha_i} \mid 1 \leq i \leq k \}$$

is a finite subcollection such that the union of the elements of $\mathcal{W}_0$ contains $U_x \times Y$. $\square$

It is also not hard to see that compactness is necessary:

Example 14.10. *Let $X = Y = \mathbb{R}$.*

Let

$$\mathcal{W} = \{ W_n = (-1/n, 1/n) \times (-n, n) \mid n \in \mathbb{N} \}.$$

Then $\mathcal{W}$ is a collection of open sets and every point of the y -axis is contained in an element of $\mathcal{W}$.

Let U be a neighbourhood of 0. Then we may find $a > 0$ such that

$$0 \in (-a, a) \subset U.$$

Pick

$$n \quad \text{such that} \quad \frac{1}{n} > \frac{a}{2}.$$

then

$$\frac{a}{2} \in U$$

but the union of the elements of $\mathcal{W}$ does not contain

$$\left(\frac{a}{2}, n\right).$$

Thus the union does not contain any set of the form

$$U \times Y$$

where U is a neighbourhood of zero.

Proof of (14.7). Suppose that $X \times Y$ is compact. As the projection maps $p: X \times Y \rightarrow X$ and $q: X \times Y \rightarrow Y$ are both continuous and surjective it follows that X and Y are compact.

Now suppose that both X and Y are compact. Let

$$\mathcal{W} = \{ W_\alpha \mid \alpha \in \Lambda \}$$

be an open cover of $X \times Y$. As Y is compact (14.9) implies that for every $x \in X$ we may find U_x such that $U_x \times Y$ is contained in the union of finitely many elements of $\mathcal{W}$.

It follows that we have an open cover

$$\mathcal{U} = \{ U_x \mid x \in X \}$$

of X . As X is compact there is a finite subcover,

$$\mathcal{U} = \{ U_{x_i} \mid 1 \leq i \leq k \}.$$

By assumption, for each U_{x_i} we may find finitely many elements $\mathcal{W}_i$ of $\mathcal{W}$ whose union contains $U_{x_i} \times Y$.

$$\mathcal{W}_0 = \bigcup_{i=1}^k \mathcal{W}_i$$

is then a finite subcover of $X \times Y$, so that $X \times Y$ is compact. □