

12. PATH-CONNECTED

Definition-Lemma 12.1. *Let X be a topological space.*

*A **path** from $a \in X$ to $b \in X$ is a continuous function*

$$f: [0, 1] \longrightarrow X$$

such that $f(0) = a$ and $f(1) = b$.

*We say that X is **path-connected** if there is path between any two points a and $b \in X$.*

The relation $\sim$ on $X \times X$ given by $x \sim y$ if and only if there is a path from x to y is an equivalence relation.

*The equivalence classes of this equivalence relation are called the **path components**.*

Proof. We have to show $\sim$ is reflexive, symmetric and transitive.

Suppose that $x \in X$. Then the constant function

$$f: [0, 1] \longrightarrow X \quad \text{given by} \quad f(t) = x$$

is continuous. $x = f(0) = f(1)$ and so there is a path from x to x . Thus $x \sim x$ and $\sim$ is reflexive.

Now suppose that $x \sim y$. Then we may find a path from x to y , a continuous function

$$f: [0, 1] \longrightarrow X$$

such that $f(0) = x$ and $f(1) = y$. Define a function

$$g: [0, 1] \longrightarrow X \quad \text{by the rule} \quad g(t) = f(1 - t).$$

Then g is continuous as it is the composition of the continuous function

$$[0, 1] \longrightarrow [0, 1] \quad \text{given by} \quad t \longrightarrow 1 - t$$

and the function f . On the other hand,

$$g(0) = f(1) = y \quad \text{and} \quad g(1) = f(0) = x.$$

Thus g is a path from y to x . Hence $y \sim x$ and so $\sim$ is symmetric.

Now suppose that $x \sim y$ and $y \sim z$. Let

$$f: [0, 1] \longrightarrow X \quad \text{and} \quad g: [0, 1] \longrightarrow X$$

be paths from x to y and y to z . Let

$$f_1: [0, 1/2] \longrightarrow X$$

be the function given by

$$f_1(t) = f(2t).$$

Then f_1 is continuous as it is the composition of the continuous function

$$[0, 1/2] \longrightarrow [0, 1] \quad \text{given by} \quad t \longrightarrow 2t$$

and the function f . Let

$$g_1: [1/2, 1] \longrightarrow X$$

be the function given by

$$g_1(t) = g(2t - 1).$$

Then g_1 is continuous as it is the composition of the continuous function

$$[1/2, 1] \longrightarrow [0, 1] \quad \text{given by} \quad t \longrightarrow 2t - 1$$

and the function g . Note that

$$[0, 1/2] \cap [1/2, 1] = [1/2, 1/2]$$

and that

$$\begin{aligned} f_1(1/2) &= f(1) \\ &= y \\ &= g(0) \\ &= g_1(1/2). \end{aligned}$$

Thus we may patch together the two functions f_1 and g_1 to get a continuous function

$$h: [0, 1] \longrightarrow X$$

given by

$$h(t) = \begin{cases} f(2t) & \text{if } t \in [0, 1/2] \\ g(2t - 1) & \text{otherwise.} \end{cases}$$

As $h(0) = f(0) = x$ and $h(1) = g(1) = z$ then h is a path from x to z . Thus $x \sim z$ and $\sim$ is transitive.

Thus $\sim$ is an equivalence relation. □

Example 12.2. *Every open ball in $\mathbb{R}^n$ is path-connected.*

Suppose that we have the open ball $U = B_r(a)$ of radius $r > 0$ centred at a . By (12.1) it is enough to show that there is a path from a to any other point $b \in U$. Let

$$f: [0, 1] \longrightarrow X$$

be the function given as

$$f(t) = ta + (1 - t)b.$$

Here we consider a and b as vectors in $\mathbb{R}^n$. If we write out f in co-ordinates we get a function whose components are linear polynomials. Thus f is continuous. $f(0) = a$ and $f(1) = b$ and so f is a path from a to b .

There is a connection between the two notions of connectedness:

Lemma 12.3. *Every path-connected topological space is connected.*

Proof. Let X be a path-connected topological space. Suppose that

$$g: X \longrightarrow \{0, 1\}$$

is a continuous function. Let a and b be any two points of X . Pick a path from a to b , a continuous function

$$f: [0, 1] \longrightarrow X$$

such that $f(0) = a$ and $f(1) = b$. The composition of f and g is a continuous function

$$h = g \circ f: [0, 1] \longrightarrow \{0, 1\}.$$

As $[0, 1]$ is connected, h is constant. But then

$$\begin{aligned} g(a) &= g(f(0)) \\ &= h(0) \\ &= h(1) \\ &= g(f(1)) \\ &= g(b). \end{aligned}$$

Thus g is constant. But then X is connected. $\square$

It follows that every open ball in $\mathbb{R}^n$ is connected as every open ball path is path-connected.

Example 12.4. *Let B be the graph of the function*

$$f: (0, \infty) \longrightarrow \mathbb{R} \quad \text{given by} \quad f(x) = \sin\left(\frac{1}{x}\right).$$

Note that f is the composition of the rational function

$$(0, \infty) \longrightarrow \mathbb{R} \quad \text{given by} \quad x \longrightarrow \frac{1}{x},$$

which is continuous, and the continuous function

$$\sin: \mathbb{R} \longrightarrow \mathbb{R} \quad \text{given by} \quad x \longrightarrow \sin x.$$

Thus f is continuous. It follows that the graph

$$B = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x \in (0, \infty) \right\}$$

is connected. In fact B is also path-connected.

Consider the C closure of B . B is contained in the box (open at one side) given by the lines

$$x = 0, \quad y = -1 \quad \text{and} \quad y = 1.$$

This is a closed set and so C is contained in this box.

Now consider a slightly smaller box,

$$x = a, \quad y = -1 \quad \text{and} \quad y = 1,$$

where $a > 0$. The intersection of B with this smaller box is the graph of the function $\sin$ on the interval $[a, \infty)$. We will see later that the graph of a continuous function on a closed interval is closed.

Thus the accumulation points of B are constrained to lie in the set

$$A = \{ (0, y) \mid -1 \leq y \leq 1 \},$$

the line interval between $(0, -1)$ and $(0, 1)$. Note that A is closed (it is the product of the two closed sets $[0, 0]$ and $[-1, 1]$).

Now $\sin$ takes on every value between -1 and 1 (by the IVT). Pick t such that $\sin t = y \in [-1, 1]$. Then

$$\sin(t + 2m\pi) = y,$$

by periodicity. If

$$a_m = \frac{1}{t + 2m\pi}$$

then

$$f(a_m) = y$$

and so

$$(a_m, y) \in B.$$

0 is an accumulation point of the set

$$\{ a_m \mid m \in \mathbb{N} \}.$$

Thus $(0, y)$ is an accumulation point of the set

$$\{ (a_m, y) \mid m \in \mathbb{N} \}.$$

It follows that the closure of B contains A . Putting all of this together we have

$$X = \overline{B} = A \cup B.$$

X is connected, as the closure of a connected set is connected.

Let

$$g: [0, 1] \longrightarrow X$$

be a path, starting in A . Consider the set $g^{-1}(A) \subset [0, 1]$. As A is closed, it follows that $G = g^{-1}(A)$ is closed. Let s be the supremum of G . As G is closed $s \in G$. But then $f(s) \in A$. Then g induces a continuous function

$$g: [s, 1] \longrightarrow X$$

Reparametrising, we may arrange that $s = 0$, so that we get a path from $a = g(0)$ to a point $b = g(1)$ not in B . In fact, we have arranged that

$$g(0, 1] \subset B.$$

By hypothesis,

$$g(0) = (0, y) \quad \text{and} \quad g(1) = (a, \sin(a)) \in B.$$

The function $p \circ g$, where p is projection onto the first factor, is continuous. By the IVT we may find t_m such that

$$p \circ g(t_m) = \frac{1}{\pi/2 + 2m\pi}$$

for any m such that

$$\frac{1}{\pi/2 + 2m\pi} < a.$$

Similarly we may find points s_m such that

$$p \circ g(s_m) = \frac{1}{3\pi/2 + 2m\pi}.$$

for any m sufficiently large. In this case

$$q \circ g(t_m) = \sin(\pi/2 + 2m\pi) = 1 \quad \text{and} \quad q \circ g(s_m) = \sin(3\pi/2 + 2m\pi) = -1,$$

where q is projection onto the second factor. The points t_m and s_m approach 0. This contradicts the fact that

$$q \circ g$$

is a continuous function. Thus there is no such path.

As A and B are path-connected, it follows that the path components of X are A and B .

Definition-Lemma 12.5. *Let X be a topological space. Define a relation $\sim$ on $X \times X$ by the rule $x \sim y$ if and only if x and y belong to a connected (in the subspace topology) subset of X .*

*$\sim$ is an equivalence relation. The equivalence classes are called the **connected components** of X .*

Proof. We have to show $\sim$ is reflexive, symmetric and transitive.

Suppose that $x \in X$. The set $\{x\}$ is connected and $x \in \{x\}$. Thus $x \sim x$ and $\sim$ is reflexive.

Now suppose that $x \sim y$. Then there a connected subset C of X such that $x \in C$ and $y \in C$. But then $y \in C$ and $x \in C$ and C is a connected subset of X . Hence $y \sim x$ and so $\sim$ is symmetric.

Now suppose that $x \sim y$ and $y \sim z$. Then we may find connected subsets A and B of X such that x and $y \in A$ and y and $z \in B$. Let $C = A \cup B$. Let

$$f: C \longrightarrow \{0, 1\}$$

be a continuous function. Then $f|_A$ and $f|_B$ are continuous functions and so they are constant, as A and B are connected. But $y \in A \cap B$ and so f is constant. x and $z \in C$, and C is connected. Thus $x \sim z$ and $\sim$ is transitive.

Thus $\sim$ is an equivalence relation. $\square$

Note that one way to think of the connected component of X containing $x \in X$, that is, the equivalence class generated by x , is that it is the maximal connected subspace of X containing x . In particular the connected components are closed, since if A is a connected subspace of X containing x then so is $\bar{A}$.

Note that the connected components are unions of the path components. Here is a simple criteria to guarantee that connected and path-connected are equivalent:

Definition-Lemma 12.6. *Let X be a topological space. We say that X is **locally path-connected** if every point has a neighbourhood which is path-connected.*

If X is locally path-connected then the connected components and the path components coincide.

Proof. Let P be a path component. Let $x \in P$ be a point of P . By assumption we may find a path connected neighbourhood V of x . As $x \in P \cap V$ it follows that $V \subset P$. But then P is open.

Let C be a connected component of X . Then C is partitioned into path components P_α , as every path component is connected. If there are two or more such path components, then C may be partitioned into two open sets,

$$P_\beta \quad \text{and} \quad \bigcup_{\alpha \neq \beta} P_\alpha$$

(here β is any fixed index and we partition C into two pieces, P_β and everything else).

This contradicts the fact that C is connected. But then C is a path component. $\square$

Note that any open set in $\mathbb{R}^n$ is locally path-connected. Indeed, given any point a of an open set U , we may find an open ball $V = B_r(a)$ contained in U . V is a neighbourhood of a and V is path-connected. Thus if we break U into its connected components then these are the path components.