

10. CANTOR'S QUESTION

This lecture forms something of a collection of parenthetical remarks and could almost go anywhere in the course.

It is quite well known that our understanding of continuous functions as they appear in analysis underwent some dramatic changes at the end of the 19th century. It was gradually realised that the naive way of thinking and drawing continuous functions is quite wrong. Most functions are far from continuous and even if they are they need not behave particularly well. The modern approach to analysis is an attempt to lay down some foundations to deal with this.

What is perhaps not quite so well known is that something similar happened in our understanding of the real numbers and its topology. It is well known that Cantor made some very important contributions to set theory but it is perhaps not so quite well known that he did so in an attempt to understand the topology of the reals.

We start by introducing a sequence of sets. We start with

$$Y_0 = \emptyset.$$

Next we put

$$Y_1 = \{0\}.$$

The next set in the sequence might come as a bit of a surprise

$$Y_2 = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$

In fact Y_1 is the set of accumulation points of Y_2 and of course Y_0 is then the set of accumulation points of Y_1 . Cantor wondered how far one can go in the construction of such subsets of $\mathbb{R}$. Is there a set Y_3 whose set of accumulation points is the set Y_2 ?

From now on we always want Y_n to start at 1. To give us some space, we are going to forget about the first point of Y_2 . The second point of Y_2 is $1/2 \in Y_2$. It is easy to make this an accumulation point, take the set Y_2 and add 1. It is then better to rescale and make the jumps shorter.

$$\left\{ \frac{1}{2} + \frac{1}{2m} = \frac{1}{2} \left(1 + \frac{1}{m} \right) \mid m \in \mathbb{N} \right\}.$$

The accumulation points of

$$\left\{ \frac{1}{3} + \frac{1}{3m} = \frac{1}{3} \left(1 + \frac{1}{m} \right) \mid m \in \mathbb{N} \right\}$$

is then just $1/3$. Continuing in this way the accumulation points of

$$\left\{ \frac{1}{n} + \frac{1}{mn} = \frac{1}{n} \left(1 + \frac{1}{m} \right) \mid m \in \mathbb{N} \right\}$$

is $1/n$. Thus the accumulation points of

$$Y_3 = \left\{ \frac{1}{n} + \frac{1}{mn} = \frac{1}{n} \left(1 + \frac{1}{m} \right) \mid m \in \mathbb{N}, n > 1 \right\}$$

is Y_2 (excluding 1). The next step is then perhaps not so surprising. If we put

$$Y_4 = \left\{ \frac{1}{n} + \frac{1}{mn} + \frac{1}{lmn} = \frac{1}{n} \left(1 + \frac{1}{m} \left(1 + \frac{1}{l} \right) \right) \mid l, m, n \in \mathbb{N}, n > 1, m > 1 \right\}$$

then the set of accumulation points of Y_4 is Y_3 (excluding some insignificant points right at the start). If we use the shorthand

$$x = \frac{1}{n} + \frac{1}{mn} + \frac{1}{lmn} + \frac{1}{klmn} = \frac{1}{n} \left(1 + \frac{1}{m} \left(1 + \frac{1}{l} \left(1 + \frac{1}{k} \right) \right) \right)$$

and put

$$Y_5 = \{ x \mid k, l, m, n \in \mathbb{N}, l > 1, m > 1, n > 1 \}$$

then the set of accumulation points of Y_5 is Y_4 (minus some initial points). In general then if we put

$$Y_{n+1} = \left\{ a \left(1 + \frac{1}{j} \right) \mid a \in Y_n, a \leq \frac{1}{2}, j \in \mathbb{N} \right\}$$

then the set of accumulation points of Y_{n+1} is Y_n (minus some initial fluff).

The genius of Cantor is that he realised that this is just the tip of the iceberg. Let

$$Y_\infty = \bigcup_n Y_n.$$

Thus Y_∞ is the union of everything that came before. Of course if we take the limit points of Y_∞ then we must get Y_n , minus some initial points, since Y_∞ contains Y_{n+1} .

Let

$$Y_{\infty+1} = \left\{ a \left(1 + \frac{1}{n^2} \right) \mid n \in \mathbb{N}, a \in Y_\infty, a \leq \frac{1}{2} \right\}.$$

The set of accumulation points of $Y_{\infty+1}$ is Y_∞ , minus some initial points, but $Y_{\infty+1}$ is different from Y_∞ (this is where the square comes in. This gives us different fractions than before). We use $\frac{1}{n^2}$ to give ourselves some more room. It is clear that we can keep going from here. It is expedient at this point to change notation and replace ∞ by ω . It also expedient to stop keeping track of the exact definition of Y_α .

It is clear that we can construct

$$Y_{\omega+2} \quad Y_{\omega+3}, \quad Y_{\omega+n}, \quad \dots$$

If we take the union of everything that came before then we get a new set.

$$Y_{\omega 2} = \bigcup_n Y_{\omega+n}.$$

To go any further, it is time to introduce the notion of an ordinal. We use von Neumann's definition. The first ordinal is

$$0 = \emptyset.$$

If we are given an ordinal α , then we can create a **successor ordinal**

$$\alpha^+ = \alpha \cup \{ \alpha \}.$$

Let's consider this more carefully. The elements of α^+ , are all of the elements of α plus α itself (in fact given any set x we can always construct the set x^+).

$$1 = 0^+ = \{ 0 \} = \{ \emptyset \}.$$

We can keep going

$$2 = 1^+ = \{ 0, 1 \} = \{ \emptyset, \{ \emptyset \} \}.$$

From here, we have

$$3 = 2^+ = \{ 0, 1, 2 \}.$$

Continuing in this way we can create all finite ordinals. Note that n has the pleasing quality that it has n elements (if you see what I mean).

The next step is to take the union of everything we had before

$$\omega = \bigcup_n n = \{ 0, 1, 2, 3, \dots, n, \dots, \}.$$

ω is called a **limit ordinal**, it is the union of everything before. Note that ω is not the successor of any ordinal. We keep going

$$\omega + 1 = \omega^+ = \{ 0, 1, 2, 3, \dots, n, \dots, \omega \}.$$

From here we construct

$$\omega + 2 = (\omega + 1)^+ = \{ 0, 1, 2, 3, \dots, n, \dots, \omega, \omega + 1 \}.$$

If we keep going and take the union we get

$$\omega 2 = \bigcup_n (\omega + n) = \{ 0, 1, 2, 3, \dots, n, \dots, \omega, \omega + 1, \omega + 2, \omega + 3, \dots, \}.$$

There is no reason to stop here

$$\omega(n + 1) = \bigcup_{3_m} \omega n + m.$$

From there we can construct

$$\omega^2 = \bigcup_n \omega n.$$

We can construct $\omega^3, \omega^4, \dots$ and then take the limit

$$\omega^\omega = \bigcup_n \omega^n.$$

Then we can construct

$$\omega^{\omega^\omega}.$$

If we keep going we can construct ω raised to the power ω , ω times. We can add and multiply ordinals

$$\omega^\omega 3 + \omega^7 2 + 3,$$

and so on.

We are now almost ready to answer Cantor's question. Any ordinal α we get in this fashion can be realised in terms of Y_α . Now we have the definition of an ordinal, following Cantor, we can better state Cantor's question. Let X be a topological space. Define a sequence of spaces X^α as follows.

$$X^\alpha = \begin{cases} X & \text{if } \alpha = 0 \\ \overline{X^\beta} & \text{if } \alpha = \beta + 1 \text{ is a successor ordinal} \\ \bigcap_{\beta < \alpha} X_\beta & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

This is an instance of transfinite induction. The smallest ordinal such that

$$X^{\alpha+1} = X^\alpha$$

is then called the **Cantor-Bendixson rank** of X .

Question 10.1 (Cantor). *What is the largest ordinal α such we can write down a subset Y_α whose Cantor-Bendixson rank is α , such that*

$$Y_\alpha^\alpha = \emptyset.$$

Suppose we take every ordinal we get this way and take the limit. We get a new ordinal ω_1 . It is not possible to realise ω_1 as one of the Y_α . We can describe ω_1 in a slightly different way. An ordinal is well-ordered. This means there is an ordering ($\alpha < \beta$ just means $\alpha \in \beta$) and every set of ordinals contains a least element. In fact we can prove that an ordinal is well-ordered by ordinal induction. One can prove that if α is well-ordered then α^+ is well-ordered and one can prove that any limit ordinal is well-ordered.

Consider all ways to order a countable set. There is ω , $\omega + 1$ and so on (every ordinal we constructed above and much more). The set of all possible ways to order a countable set is an ordinal ω_1 .

Definition 10.2. *The set of all countable ordinals is an ordinal ω_1 .*

Almost by definition, ω_1 is not countable. It is the first uncountable ordinal.

There is another way to think of all of this. One can think of Cantor's ordinal question as asking what is the biggest ordinal that can be embedded into the reals. In other words, the ordering on the ordinal coincides with the ordering it inherits from the reals.

Somewhat amazingly, any countable ordinal sits in the reals.

Definition 10.3. *A **totally ordered** set is a pair $(X, <)$, where X is a set and $<$ is a relation with the following properties:*

- (1) *If $x \neq y$ then either $x < y$ or $y < x$.*
- (2) *$x < x$ does not hold.*
- (3) *If $x < y$ and $y < z$ then $x < z$.*

Definition 10.4. *A **well ordered** set is a totally order set $(X, <)$ such that every subset $Y \subset X$ has a **minimal** element, an element $m \in Y$ such that if $y \in Y$ then $m \leq y$.*

Definition-Theorem 10.5 (Mostowski's Collapsing Theorem). *Every well-ordered set is equivalent to an ordinal $(\alpha, <)$, that is, there is a bijection*

$$f: X \longrightarrow \alpha,$$

that preserves the order, $x < y$ if and only if $f(x) < f(y)$.

*α is called the **order type** of $(X, <)$.*

Definition 10.6. *Let $(X, <)$ be a totally ordered set. The **order topology** on X is the topology generated by the half rays*

$$\{x \in X \mid x < a\} \quad \text{and} \quad \{x \in X \mid x > a\}$$

for all $a \in X$.

Note that to say the half rays generate the order topology means they are a subbasis. If X has at least two elements, then the order topology is the topology with the basis given by the open intervals

$$(a, b) = \{x \in X \mid a < x < b\}$$

and the half rays.

Note that the order topology on $\mathbb{R}$ coincides with the Euclidean topology, simply because a basis for the order topology includes the open intervals and the half rays are open in the Euclidean topology.

If we are given an ordinal and give the ordinal a topology using the order topology then the limit ordinals are the accumulation points in the order topology.

Definition 10.7. *We say that a continuous function $f: X \longrightarrow Y$ is an **embedding** if f is injective and it induces a homeomorphism with its image.*

Theorem 10.8 (Cantor). *Let α be an ordinal.*

Then there is an embedding of α into $\mathbb{R}$, where we give both sets the order topology, if and only if α is countable.

Equivalently, there is a subset $Z_\alpha \subset \mathbb{R}$ such that Z_α is well-ordered, in the natural order, and the order type of Z_α is α .

In terms of the sets we defined right at the start, we would have to change notation, very slightly. Put

$$Z_1 = \{1\} \quad \text{and} \quad Z_2 = \{1, \frac{1}{2}\}$$

and continue in this way, renumbering (relabelling?) as appropriate. Thus

$$Z_n = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\}$$

and

$$Z_\omega = Y_2.$$

In terms of all of the ordinals less α less than ω_1 , all countable ordinals, this is not much of a change.

Also note that the elements of Z_α are decreasing. Either switch the ordering on the reals or Z_α (or what comes to the same thing, multiply the elements of Z_α by -1 and add one, that is, apply the transformation $x \longrightarrow 1 - x$).