

1. WHAT IS TOPOLOGY?

Analysis (or calculus) concerns itself with understanding real functions. The inputs and outputs to functions are real numbers. For example

$$\pi = 4 \tan^{-1}(1).$$

If we try to represent π as a real number then it has an infinite decimal expansion,

$$\pi = 3.141592653589793 \dots$$

Analysis and calculus have a definite quantitative approach to the study of functions. Using Taylor series, Newton-Raphson approximation and so on, we are supposed to be able to compute our functions to arbitrary precision, even if we cannot compute precisely.

One of the most important properties of a function is continuity. The definition of continuity is accordingly quantitative:

A real function f defined on an interval containing the point x_0 is continuous at x_0 if for every $\epsilon > 0$ there is a $\delta > 0$ such that if

$$|x - x_0| < \delta \quad \text{then} \quad |f(x) - f(x_0)| < \epsilon.$$

However this formal definition hides our intuitive picture of a continuous function (or perhaps better, of a discontinuous function). A function is continuous if we can draw it without taking our hand off the paper, that is, if there are no jumps or gaps. This intuitive picture has a qualitative aspect, since we are not worried about the size of the jumps. We just care if there are any gaps or if there are no gaps at all.

Some of the most striking results in calculus are the results centred on the divergence theorem. Let

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

be a function. Suppose that f is continuously second differentiable (that is, its second partial derivatives exist and they are continuous). Let V be a volume in $\mathbb{R}^3$, so that V is contained in a sphere (it is bounded), and its boundary S is piecewise smooth. Then

$$\iiint_V \nabla \cdot f \, dV = \oiint_S f \cdot \hat{n} \, dS.$$

Here $\hat{n}$ is the outward pointing normal. For example, suppose that the divergence of f is zero,

$$\nabla \cdot f = 0.$$

Then the formula above reduces to

$$\oiint_S f \cdot \hat{n} \, dS = 0.$$

The integral of $f \cdot \hat{n}$ around the boundary S of a volume V is zero, provided the divergence is zero.

Notice that this formula has a qualitative aspect. We don't say that the integral is close to zero, we say that the integral is zero. The formula also has an appealing intuitive physical interpretation. If we imagine that f represents fluid flow, then the condition that the divergence is zero may be interpreted as the statement that fluid in V is not been pumped in or removed. If this is true then the total amount of fluid in V is constant, so that the total amount of fluid crossing the boundary S must be zero.

However all of this hides some quite subtle mathematical issues. What do we mean by the inside of V ? To illustrate that this is much more complicated than it might first appear, let's drop the dimension by one:

Theorem 1.1 (Jordan Curve Theorem). *Let*

$$f: [0, 1] \longrightarrow \mathbb{R}^2$$

be a continuous function such that

$$f(a) = f(b) \quad \text{if and only if} \quad \text{either } a = b \text{ or } \{a, b\} = \{0, 1\}.$$

Then

$$\mathbb{R}^2 \setminus C$$

is divided into two regions, a bounded region and an unbounded region, where C is the image of f .

This theorem seems so obvious, it hardly seems worth calling it a theorem. If you draw a closed loop in the plane (that does not intersect itself) then this loop divides the plane into two regions (let's call the two regions the inside and the outside). However it is surprisingly hard to prove the Jordan curve theorem.

It is important to realise that this does not have anything to do with issues around the fact that continuous functions are quite complicated. In fact there is no harm in assuming that that C is a subset of the grid:

$$\{ (x, y) \in \mathbb{R}^2 \mid \text{either } x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \}.$$

In this form it starts to become clearer why this this theorem is not as easy as it looks. For example, it can be quite hard to navigate through a maze. If you try to solve the maze by picking a point inside the maze (with the intention of finding the entrance and the exit), it might not so obvious if you are in the maze or outside it.

After a little thought, one realises that we can characterise the inside by the number of times we cross the curve C starting from the outside. If we cross an odd number of times then we are on the inside.

So pick a point p very far out. This point must be on the outside. Consider a point q . Draw the line l connecting p to q and count the number of times l meets C (this is particularly straightforward if C is a grid, in which case we may assume that p is so far diagonally out that the line is neither horizontal nor vertical. We then need to make the line bend around corners). If this number is odd then q is on the inside. This seems quite routine until one realises one needs to check that if we pick another point p' on the outside then we still get that q is on the inside. This is not so obvious.

Notice that the Jordan curve theorem has a qualitative aspect (there are two regions, an inside and an outside). Topology concerns itself with the qualitative notions that come out from analysis. For example, the intermediate value theorem states that if

$$f: [a, b] \longrightarrow \mathbb{R}$$

is a continuous function then f takes on every value between $f(a)$ and $f(b)$. Similarly the extreme value theorem says that f attains its maximum on $[a, b]$.