

HOMEWORK 9, DUE WEDNESDAY MARCH 12TH

1. Show that an arbitrary product of Hausdorff spaces is Hausdorff.
2. Show that an arbitrary product of connected sets is connected.

We will call a continuous and surjective function $f: X \rightarrow Y$ a **quotient map** if there is an equivalence relation $\sim$ on X such that f satisfies the universal property of the quotient, that is, given a continuous map $g: X \rightarrow Z$ such that if

$$x \sim y \quad \text{then} \quad g(x) = g(y)$$

then there is a unique continuous function $h: Z \rightarrow Y$ which makes the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ f \downarrow & \nearrow h & \\ Y & & \end{array}$$

3. Let $f: X \rightarrow Y$ be a continuous and surjective function.
 - (i) Show that f is a quotient map if and only if Y has the finest topology such that f is continuous (that is, $V \subset Y$ is open if and only if $U = f^{-1}(V)$ is open).
 - (ii) Show that if f is open then f is a quotient map.
 - (iii) Show that if f is closed then f is a quotient map.
 - (iv) Show that if X is compact and Y is Hausdorff then f is a quotient map.
 - (v) Show that if f has a right inverse, that is, there is a continuous function $g: Y \rightarrow X$ such that $f \circ g: Y \rightarrow Y$ is the identity, then f is a quotient map.
 - (vi) Show that if $p: X \times Y \rightarrow Y$ is the projection then p is a quotient map.
 - (vii) Let X be a topological space and let $Y \subset X$ be a subspace. We say that r is a **retraction** of X onto Y if $r: X \rightarrow Y$ is a continuous map whose restriction to Y is the identity, $r|_Y = i_Y$. If $r: X \rightarrow Y$ is a retraction then show that r is a quotient map.
 - (viii) Show that the composition of quotient maps is a quotient map.
4. Let

$$X = \{ (x, y) \mid \text{either } x \geq 0 \text{ or } y = 0 \} \subset \mathbb{R}^2.$$

Let $p: \mathbb{R}^2 \longrightarrow \mathbb{R}$ denote projection onto the first factor and let

$$f: X \longrightarrow \mathbb{R}$$

be the restriction of p to X .

Show that f is a quotient map, even though f is neither open nor closed.

5. Consider the Möbius strip X_1 and the closed unit ball X_2 in $\mathbb{R}^2$. The boundary of both is homeomorphic to S^1 (in fact the boundary of the closed unit ball is S^1). Define a topological space X by taking the quotient of the disjoint union of X_1 and X_2 , modulo an equivalence relation that identifies the two boundaries (in other words, pick a homeomorphism between the two boundaries and identify corresponding points).

Show that X is homeomorphic to the real projective plane.

Challenge Problems: (Just for fun)

6. If M is an n -manifold then let

$$\lambda(M)$$

be the smallest number of open sets that are homeomorphic to $\mathbb{R}^n$ and which cover X .

What are $\lambda(S^1)$, $\lambda(S^2)$, $\lambda(S^n)$, $\lambda(S^1 \times S^1)$ and $\lambda(\mathbb{RP}^2)$?