

HOMEWORK 8, DUE WEDNESDAY MARCH 5TH

1. Show that if X is a Hausdorff topological space and A and B are compact then $A \cap B$ is compact.
2. Let

$$f: X \longrightarrow Y$$

be a function with graph

$$\Gamma_f = \{ (x, f(x)) \mid x \in X \}$$

Suppose that X and Y are topological spaces.

- (i) If Y is Hausdorff and f is continuous then show that Γ_f is closed in $X \times Y$.
- (ii) Show that if Y is compact then

$$p: X \times Y \longrightarrow X$$

is a closed map (that is, the image of a closed set is closed).

- (iii) Show that if Y is compact and Γ_f is closed in $X \times Y$ then f is continuous.

3. Let A and B be subspaces of the topological spaces X and Y . Suppose that

$$A \times B \subset W \subset X \times Y.$$

If A and B are compact and W is open then show that we can find U and V open in X and Y such that

$$A \times B \subset U \times V \subset W.$$

4. Let f and g be two functions

$$f: X \longrightarrow Y \quad \text{and} \quad g: X \longrightarrow Y.$$

Let

$$h: X \longrightarrow Y \times Y$$

be the function

$$h(x) = (f(x), g(x)).$$

- (i) Show that $f = g$ if and only if the image of h lands in the diagonal

$$\Delta = \{ (y, y) \mid y \in Y \}.$$

- (ii) Now suppose that f and g are continuous functions and Y is Hausdorff. If $A \subset X$ is dense and $f|_A = g|_A$ then show that $f = g$.

5. Let X be a Hausdorff topological space.

(i) If A and B are disjoint compact subspaces of X then show that we can find disjoint open sets U and V such that

$$A \subset U \quad \text{and} \quad B \subset V.$$

(ii) Suppose in addition that X is compact. Suppose we are given a collection of connected closed subsets

$$\{F_\alpha \mid \alpha \in \Lambda\}.$$

which are totally ordered by inclusion, so that given α and $\beta \in \Lambda$, either

$$F_\alpha \subset F_\beta \quad \text{or} \quad F_\beta \subset F_\alpha.$$

Show that

$$\bigcap_{\alpha} F_\alpha$$

is closed and connected.

Challenge Problems: (Just for fun)

6. Show that a topological space Y is compact if and only if for every topological space X the projection map

$$p: X \times Y \longrightarrow Y$$

is closed.