

HOMEWORK 7, DUE WEDNESDAY FEBRUARY 26TH

1. Let X be the topological space whose closed sets are the finite sets plus the whole of X . Show that X is compact.
2. Let X be a topological space. Show that X is compact if and only if for every collection of closed sets

$$\mathcal{F} = \{ F_\alpha \mid \alpha \in \Lambda \}$$

such that every finite subcollection

$$\mathcal{F}_0 = \{ F_\beta \mid \beta \in M \}$$

has non-empty intersection then the intersection of every element of $\mathcal{F}$ is non-empty.

3. True or false? If true then give a proof and if false then give a counterexample.

(i) If X is a compact topological space and $f: X \rightarrow \mathbb{R}$ is continuous then f achieves its maximum and minimum value.

(ii) Let $f: X \rightarrow Y$ be continuous and injective. If Y is Hausdorff then X is Hausdorff.

(iii) If A and B are compact subspaces of a topological space X then $A \cup B$ is compact.

(iv) If A and B are compact subspaces of a topological space X then $A \cap B$ is compact.

4. Let X be a compact topological space and let Y be a Hausdorff topological space. If $f: X \rightarrow Y$ is continuous and a bijection then show that f is a homeomorphism.

Challenge Problems: (Just for fun)

5. Show that one cannot find uncountably many pairwise disjoint copies (up to homeomorphism) of the letter T in $\mathbb{R}^2$.

Conclude that if $\mathbb{R}^2$ and $\mathbb{R}^n$ are homeomorphic then $n = 2$.

6. We say that a topological space X is **Noetherian** if the set of closed subsets satisfies the DCC (descending chain condition). This means that every infinite decreasing sequence of closed sets

$$F_1 \supset F_2 \supset \cdots \supset F_n \cdots$$

eventually stabilises, that is, there is an index k such that if i and $j > k$ then $F_i = F_j$.

- (i) Let X be the topological space whose closed sets are the finite sets plus the whole of X . Show that X is Noetherian.
- (ii) Show that any Noetherian topological space is compact.
- (iii) Let k be a field. Show that k^n , with the Zariski topology, is Noetherian.
- (iv) Let P be a property of topological spaces satisfying:
 If X is any topological space such that $P(Y)$ holds for all subspaces, other than $Y = X$, then $P(X)$ holds.
 Show that $P(X)$ holds for all Noetherian topological spaces.
 This is called **Noetherian induction**.
- (v) Show that every Noetherian topological space decomposes as a finite union of irreducible subsets. Show that this decomposition is unique, up to reordering, provided it is minimal (meaning we cannot discard any subsets).