

HOMEWORK 6, DUE WEDNESDAY FEBRUARY 19TH

1. Let X be the topological space whose closed sets are the finite sets plus the whole of X .

When is X connected?

2. Let X be a topological space and let Y be a connected subspace. If

$$Y \subset Z \subset \overline{Y}$$

then prove that Z is connected.

3. Let $Y \subset \mathbb{R}^n$ be a subset.

(i) We say that Y is **convex** if the line between any two points p and q of Y is contained in Y ,

$$\{tp + (1 - t)q \mid t \in [0, 1]\} \subset Y.$$

Show that if Y is convex then it is path-connected.

(ii) We say that Y is **star-shaped** about y_0 if for any point $y \in Y$ the line connecting y_0 to y is contained in Y .

Show that if Y is star-shaped then it is path-connected.

4. True or false? If true then give a proof and if false then give a counterexample.

(i) The set

$$\mathbb{Q}^2 = \{(a, b) \mid a, b \in \mathbb{Q}\}$$

is connected.

(ii) The set

$$\mathbb{Q}^2 = \{(a, b) \mid a, b \in \mathbb{Q}\}$$

is path-connected.

(iii) The set

$$\mathbb{R}^2 \setminus \mathbb{Q}^2$$

is path-connected.

(iv) The set

$$\mathbb{R}^2 \setminus \mathbb{Q}^2$$

is connected.

(v) If $f: X \rightarrow Y$ is continuous and surjective and X is path-connected then Y is path-connected.

(vi) If X and Y are path-connected topological spaces then $X \times Y$ is path-connected.

(vii) The path components of a topological space are closed.

(viii) The connected components of a topological space are open.

5. Let X be a connected topological space. We say that a point x is a **cut point** if $X - \{x\}$ is disconnected.

(i) Let $Y \subset \mathbb{R}^2$ be the union of two closed disks that intersect at one point (so that the boundary circles are tangent). Identify the cut points.

(ii) If $f: X \rightarrow Y$ is a homeomorphism then show that $x \in X$ is a cut point if and only if $y = f(x)$ is a cut point.

(iii) Show that $[0, 1]$ and $(0, 1)$ are not homeomorphic.

(iv) Give a complete list of all intervals in $\mathbb{R}$, up to homeomorphism.

(v) Show that if $\mathbb{R}$ is homeomorphic to $\mathbb{R}^n$ then $n = 1$.

Challenge Problems: (Just for fun)

6. Let X be the topological space whose closed sets are the finite sets plus the whole of X .

When is X path-connected? (It is not clear to me that there is a simple definitive answer to this question. Treat this as an open-ended question and try to find interesting cases where you can answer the question.)

7. Let X be a topological space. We say that X is **irreducible** if whenever we have two closed sets F and G such that $X = F \cup G$ then either $X = F$ or $X = G$.

(a) Let X be the topological space whose closed sets are the finite sets plus the whole of X . When is X irreducible?

(b) Show that $\mathbb{R}^n$ is not irreducible.

(c) Let $V \subset k^n$ be a Zariski closed subset, where k is a field. Let

$$I = I(V) = \{ f \in k[x_1, x_2, \dots, x_n] \mid f(p) = 0, \forall p \in V \}.$$

Show that I is an ideal in the polynomial ring $R = k[x_1, x_2, \dots, x_n]$.

Show that V is irreducible if and only if I is a prime ideal in R .

(d) Show that any line (the zero locus of a linear polynomial) in k^2 is irreducible.