

HOMEWORK 5, DUE WEDNESDAY FEBRUARY 12TH

1. The purpose of this exercise is to build up as many of the familiar functions from analysis as we can, using just the definitions and some results from the course.

Let X and Y be topological spaces.

(i) If $b \in Y$ then show that the constant function

$$f: X \longrightarrow Y \quad \text{given by} \quad f(x) = b,$$

is continuous.

(ii) If $m \in \mathbb{R}$ is a real then show that the function

$$f: \mathbb{R} \longrightarrow \mathbb{R} \quad \text{given by} \quad f(x) = mx$$

is continuous.

(iii) Show that the function

$$a: \mathbb{R}^2 \longrightarrow \mathbb{R} \quad \text{given by} \quad a(x, y) = x + y,$$

is continuous.

(iv) If $f: X \longrightarrow \mathbb{R}$ and $g: Y \longrightarrow \mathbb{R}$ are continuous functions then show that the function

$$f + g: X \longrightarrow \mathbb{R} \quad \text{given by} \quad (f + g)(x) = f(x) + g(x)$$

is continuous.

(v) Show that the function

$$m: \mathbb{R}^2 \longrightarrow \mathbb{R} \quad \text{given by} \quad m(x, y) = xy,$$

is continuous.

(vi) If $f: X \longrightarrow \mathbb{R}$ and $g: X \longrightarrow \mathbb{R}$ are continuous functions then show that the function

$$fg: X \longrightarrow \mathbb{R} \quad \text{given by} \quad (fg)(x) = f(x)g(x)$$

is continuous.

(vii) Show that the function

$$f: \mathbb{R} \longrightarrow \mathbb{R} \quad \text{given by} \quad x \longrightarrow x^n$$

is continuous.

(viii) Any polynomial function

$$f: \mathbb{R} \longrightarrow \mathbb{R} \quad \text{given by} \quad f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

where $a_0, a_1, \dots, a_{n+1}$ are real numbers, is continuous.

(ix) The function

$$f: \mathbb{R}^* \longrightarrow \mathbb{R} \quad \text{given by} \quad f(x) = \frac{1}{x}$$

is continuous, where

$$\mathbb{R}^* = \mathbb{R} \setminus \{0\}.$$

(x) Let

$$g(x) = \sum b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0,$$

be a polynomial function, where $b_0, b_1, \dots, b_m$ are real numbers. Show that the set of points where g is non-zero is an open subset U of $\mathbb{R}$.

Show that the rational function

$$\phi: U \longrightarrow \mathbb{R} \quad \text{given by} \quad \phi(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0}$$

is continuous.

2. Let $f: X \longrightarrow Y$ be any map of sets whose image lies in a subset B of Y . Note that there is a unique function $g: X \longrightarrow B$ with the property that $f = g \circ j$, where $j: B \longrightarrow Y$ is the natural inclusion.

Now suppose that X and Y are topological spaces and give B the subspace topology.

Show that f is continuous if and only if g is continuous.

3. True or false? If true then give a proof and if false then give a counterexample.

(i) If $f: X \longrightarrow Y$ is a continuous map of topological spaces and f is a bijection then f is a homeomorphism.

(ii) If $f: X \longrightarrow Y$ is a continuous function between two topological spaces then the inverse image of every closed set is closed.

(iii) If $f: X \longrightarrow Y$ is a function between two topological spaces and the inverse image of every closed set is closed then f is continuous.

(iv) The function

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$

defined as

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{otherwise} \end{cases}$$

is continuous.

(v) If $f: X \longrightarrow Y$ is a continuous function between two topological spaces and $A \subset X$ then

$$f(\overline{A}) = \overline{f(A)}.$$

(vi) If $f: X \longrightarrow Y$ is a continuous function between two topological spaces and $A \subset X$ then

$$f(\overline{A}) \subset \overline{f(A)}.$$

4. Let

$$S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1 \}$$

be the unit sphere.

(i) Show that S^n is a closed subset of $\mathbb{R}^{n+1}$. Let

$$N = (0, 0, 0, \dots, 0, 1) \in \mathbb{R}^{n+1}.$$

Show that $N \in S^n$ (N is sometimes called the North pole). Show that $U = S^n - \{ N \}$ is an open subset of S^n . Let H be the hyperplane determined by the equation $x_{n+1} = 0$,

$$H = \{ (a_1, a_2, \dots, a_{n+1}) \mid a_{n+1} = 0. \}$$

Show that H is homeomorphic to $\mathbb{R}^n$.

(ii) Define a function

$$f: U \longrightarrow H$$

as follows. Consider the straight line connecting the north pole N to the point $P \in \mathbb{R}^{n+1} \setminus \{ N \}$. This line intersects H in a unique point Q . If $P \in U$ then we send P to this unique point Q . f is called **Stereographic projection**.

Write down a formula for f and show that f is continuous.

(iii) Show that f is a homeomorphism of U with H , so that U is homeomorphic to $\mathbb{R}^n$.

5. Let

$$Y = \bar{B}_1(0) = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}$$

be the closed unit ball, centred at the origin. Let a and b be any two points on the unit circle, the boundary of Y . Let $C_{a,b}$ be the part of the boundary going from a to b (going anti-clockwise), so that $C_{a,b}$ is an arc of the unit circle including a and b . Let

$$U_{a,b} = Y \setminus C_{a,b}.$$

Show that the homeomorphism type of $U_{a,b}$ does not depend on a and b (we even allow $a = b$). (*Hint: you may use the fact that $\cos$ and $\sin$ are continuous functions*).

Challenge Problems: (Just for fun)

6. Let $k = \mathbb{R}$ (or $\mathbb{C}$). Show that the Zariski topology is coarser than the Euclidean topology on k^n .

7. Show one direction of Cantor's theorem, show that if we can embed the ordinal α into $\mathbb{R}$ then α is countable.