

HOMEWORK 3, DUE WEDNESDAY JANUARY 29TH

1. Let $(X, \mathcal{T})$ be a topological space. If $A \subset X$ is any subset then we say that $x \in X$ is an **accumulation point** if the closure of $A \setminus \{x\}$ contains x .

Show that the closure of A is the union of A and all of its accumulation points.

2. Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$ and let $(Y, \mathcal{S})$ be a topological space with basis $\mathcal{C}$.

Show that

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

3. Let $(X, \mathcal{T})$ be a topological space. We say that $(X, \mathcal{T})$ is **Hausdorff** if for any two points $x \neq y \in X$ we may find two disjoint neighbourhoods F and G of x and y .

Show that the following are equivalent:

- (i) $(X, \mathcal{T})$ is Hausdorff.
- (ii) For any two points $x \neq y$ we can find two disjoint open subsets U and V such that $x \in U$ and $y \in V$.
- (iii) For any two points $x \neq y$ we can find a closed neighbourhood A of x not containing y (that is, $y \notin A$).
- (iv) The diagonal

$$\Delta = \{(x, x) \mid x \in X\}$$

is closed in the product topology.

4. True or false? If true then give a proof and if false then give a counterexample.

- (i) If $(X, \mathcal{T})$ is a topological space and $Y \subset X$ is a subset and $U \subset Y$ is open in the subspace topology then U is open in X .
- (ii) If $(X, \mathcal{T})$ is a Hausdorff topological space then every singleton subset $\{x\}$ is closed.
- (iii) If $(X, \mathcal{T})$ is a topological space and every singleton subset is closed then $(X, \mathcal{T})$ is Hausdorff.
- (iv) If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are Hausdorff topological spaces then the product $(X \times Y, \mathcal{R})$ is Hausdorff.
- (v) If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are two topological spaces and $A \subset X$, $B \subset Y$ then

$$\overline{A \times B} = \overline{A} \times \overline{B}$$

in the product topology on $X \times Y$.

- (vi) Every subspace of a Hausdorff topological space is Hausdorff.
 5. If (X, d) is a metric space then the induced topological space $(X, \mathcal{T})$ is Hausdorff.

Challenge Problems: (Just for fun)

6. (i) Show that the Zariski topology on k^n is in general not Hausdorff.
 (ii) Show that the Zariski topology on k^2 is not the product topology (of k with itself, with the Zariski topology).
 (iii) Show that the diagonal

$$\Delta = \{ (x, x) \mid x \in k^n \} \subset k^{2n}$$

is closed.

- (iv) If $m \leq n$ then let $k^m \longrightarrow k^n$ be the obvious inclusion given by

$$(a_1, a_2, \dots, a_m) \longrightarrow (a_1, a_2, \dots, a_m, 0, 0, \dots, 0).$$

Show that subspace topology on k^m coincides with the Zariski topology on k^m .

- (v) Conclude that the Zariski topology on k^{2n} is not the product topology (of k^n with itself, with the Zariski topology).