

HOMEWORK 2, DUE WEDNESDAY JANUARY 22ND

In question 2, 4 (iv), (v) and 5, we work with the Euclidean topology on $\mathbb{R}$.

1. Let X be a set. Give X the topology $\mathcal{T}$ where every finite set is closed, plus X . If Y is a subset of X then determine

- (i) The interior of Y .
- (ii) The closure of Y .
- (iii) The boundary of Y .

2. Let $a < b \in \mathbb{R}$ and let $Y = [a, b) \subset \mathbb{R}$. What is

- (i) The interior of Y ?
- (ii) The closure of Y ?
- (iii) The boundary of Y ?

3. Let (X, d) be a metric space, let $a \in X$ be a point of X and let $r > 0$ be a positive real. Let

$$B = \bar{B}_r(a) = \{x \in X \mid d(a, x) \leq r\}.$$

Show that B is closed in X . $\bar{B}_r(a)$ is called the **closed ball** of radius r centred about a .

4. True or false? If true then give a proof and if false then give a counterexample.

(i) Let (X, d) be a metric space, let $a \in X$ be a point of X and let $r > 0$ be a positive real. Let $Y = B_r(a)$ be the open ball of radius r centred about a . Then the closure of Y is the closed ball $\bar{B}_r(a)$ of radius r centred about a .

(ii) If $(X, \mathcal{T})$ is a topological space and $Y \subset X$ is a subset then

$$\overline{X \setminus Y} = X \setminus \text{int } Y.$$

(iii) If $(X, \mathcal{T})$ is a topological space and $Y \subset X$ and $Z \subset X$ are two subsets then

$$\text{int}(Y \cup Z) = \text{int}(Y) \cup \text{int}(Z).$$

(iv) The integers $\mathbb{Z}$ are dense in the reals $\mathbb{R}$.

(v) The rationals $\mathbb{Q}$ are dense in the reals $\mathbb{R}$.

5. Show that every open subset of $\mathbb{R}$ is a disjoint union of open intervals.

6. (i) Let $(X, \mathcal{T})$ be a topological space. Starting with any subset $Y \subset X$ (and any X) what is the maximum number of distinct subsets one can obtain by taking the closure and the complement (as many times as you please, in whatever order you please)?

Challenge Problems: (Just for fun)

(ii) Prove this.

7. Let k be a field (for example the reals or the complexes) and let n be a positive integer. Let $X = k^n$ be the set of all n -tuples of elements of k (so that X has the structure of a k -vector space). Let $k[x_1, x_2, \dots, x_n]$ be the set of all polynomials in the variables $x_1, x_2, \dots, x_n$ with coefficients in k (for example, real or complex polynomials in n variables). We say that a subset $V \subset k^n$ is a **subvariety** if V is the set of zeroes of a collection of polynomials S in $k[x_1, x_2, \dots, x_n]$,

$$V = V(S) = \{ (a_1, a_2, \dots, a_n) \in k^n \mid f(a_1, a_2, \dots, a_n) = 0, \forall f \in S \}.$$

(a) Show that there is a topology $\mathcal{T}$ on X such that the closed subsets are the subvarieties. This is called the **Zariski** topology.

(b) Describe the Zariski topology when $n = 1$, so that $X = k$?

(c) Given a polynomial $f \in k[x_1, x_2, \dots, x_n]$, let

$$U_f = \{ (a_1, a_2, \dots, a_n) \in k^n \mid f(a_1, a_2, \dots, a_n) \neq 0 \}.$$

Show that U_f is open in the Zariski topology.

(d) Show that the set

$$\mathcal{B} = \{ U_f \mid f \in k[x_1, x_2, \dots, x_n] \}$$

is a basis for the Zariski topology.

(e)* Show that every closed subset is the zero locus of finitely many polynomials (this uses some highly non-trivial algebra).

(f) Using (e), show that every open subset U is the union of finitely basis elements.