

HOMEWORK 1, DUE WEDNESDAY JANUARY 15TH

1. Find all topologies on the set

$$X = \{a, b, c\}.$$

2. Find a topology $\mathcal{T}$ on the set

$$X = \{a, b, c, d\}$$

such that a subset is open if and only if it is closed and yet $(X, \mathcal{T})$ is neither the trivial nor the discrete topology.

3. Let X be a set and define the function

$$d: X \times X \longrightarrow \mathbb{R}$$

by the rule

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

- (i) Show that (X, d) is a metric space.
 - (ii) What is the associated topology?
4. True or false? If true then give a proof and if false then give a counterexample.
 - (i) Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same set X . Then we can always compare $\mathcal{T}_1$ and $\mathcal{T}_2$ (that is, either $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ or $\mathcal{T}_1$ is finer than $\mathcal{T}_2$).
 - (ii) The topology $(X, \mathcal{T})$ associated to a metric space (X, d) is never the trivial topology.
 - (iii) If (X, d_1) and (X, d_2) are two metric spaces that give the same topology on X then $d_1 = d_2$.
 - (iv) The set $[a, b) \subset \mathbb{R}$ is open, in the Euclidean topology, where $a < b$.
 - (v) The set $(a, \infty) \subset \mathbb{R}$ is open, in the Euclidean topology.
 - (vi) Let X be any set. Then the set of all infinite subsets of X , union the empty set, is a topology on X .
 - (vii) The Euclidean topology on $\mathbb{R}$ and the topology on $\mathbb{R}$ given by the metric $d(x, y) = |x - y|$ are the same.
 5. Let $X = \mathbb{R}$ and let

$$\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R} \cup \{-\infty, \infty\}\}.$$

- (i) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$.
- (ii) Is this topology coarser or finer than the Euclidean topology?

Challenge Problems: (Just for fun)

The goal of the next couple of questions is to give a proof of the Schröder-Bernstein theorem (different from the usual one). The idea is to use the Knaster-Tarski theorem.

We first go through the proof of the Knaster-Tarski theorem. We will need a couple of definitions.

A **partially ordered set** is a pair $(X, \leq)$, where X is a set and $\leq$ is a binary relation with the following properties:

- (1) **reflexive:** $x \leq x$,
- (2) **anti-symmetric:** If $x \leq y$ and $y \leq x$ then $x = y$, and
- (3) **transitive:** If $x \leq y$ and $y \leq z$ then $x \leq z$.

A function

$$f: X \longrightarrow Y$$

between partially ordered sets is **monotonic** if

$$x_1 \leq x_2 \quad \text{implies} \quad f(x_1) \leq f(x_2).$$

A **complete lattice** is a partially ordered set $(L, \leq)$ with the property that every subset $A \subset L$ has a **supremum** (aka, least upper bound), an element $s = \vee A \in L$, such that if $a \in A$ then $a \leq s$ and if $t \in L$ has the property that $a \leq t$ for all $a \in A$ then $s \leq t$. Similarly every subset $A \subset L$ has an **infimum** (aka, greatest lower bound), an element $i = \wedge A \in L$, such that if $a \in A$ then $i \leq a$ and if $j \in L$ has the property that $j \leq a$ for all $a \in A$ then $j \leq i$.

Theorem 0.1 (Knaster-Tarski). *Let $(L, \leq)$ be a complete lattice and let $f: L \longrightarrow L$ be a monotonic map.*

Let

$$P = \{x \in L \mid f(x) = x\}$$

be the subset of fixed points. Then $(P, \leq)$ is a complete lattice.

Note that a complete lattice is always non-empty (the infimum 1_L and supremum 0_L of the empty set are elements of L). Often the most useful application of (0.1) is simply the observation that there are fixed points.

6. Let X be a set and let $L = \mathcal{O}(X)$ be the power set of X . Show that $(L, \leq)$ is a complete lattice where

$$A \leq B \quad \text{if and only if} \quad A \subset B.$$

We now turn to a proof of (0.1):

7. Let

$$D = \{x \in L \mid x \leq f(x)\}.$$

(a) Show that D is non-empty (consider the smallest element 0_L of L).

(b) Show that

$$u = \vee D$$

is a fixed point.

(c) Show that u is the largest element of P .

(d) Let $(L, \preceq)$ be the lattice obtained from $(L, \leq)$ by reversing the inequality in $(L, \leq)$. Thus

$$x \preceq y \quad \text{if and only if} \quad y \leq x.$$

Observe that $f: L \rightarrow L$ is a monotonic map of $(L, \preceq)$ and use this to show that P contains a smallest element.

(e) If a and $b \in L$ then let

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Let W be any subset of P , let $w = \vee W$ be the supremum of W (in L) and let 1_L be the largest element of L . Observe that $[w, 1_L]$ is a complete lattice and that $f([w, 1_L]) \subset [w, 1_L]$, so that f induces a monotonic map of the complete lattice $([w, 1_L], \leq)$. Use this to conclude that $W \subset P$ has a supremum in P .

We now show how to use (0.1) to prove Schröder-Bernstein.

8. Let $f: X \rightarrow Y$ and let $g: Y \rightarrow X$ be two injective functions. Our goal is find a bijection $h: X \rightarrow Y$.

(a) Let $(L, \leq) = (\mathcal{P}(X), \subset)$. Define a function

$$\phi: L \rightarrow L$$

by the rule

$$\phi(A) = X \setminus g(Y \setminus f(A)).$$

Show that ϕ is monotonic.

(b) Conclude that there is a set C such that

$$X \setminus C = g(Y \setminus f(C)).$$

(c) Use this to construct a bijection $h: X \rightarrow Y$.