

MODEL ANSWERS TO THE NINTH HOMEWORK

1. Suppose that there is a unique homotopy class of path for every pair of points p and q . Then there must be a path from p to q , so that X is certainly path connected. If we take $p = q = x_0$ then there is only one homotopy class of loop based at x_0 , so that the fundamental group is trivial.

Now suppose that X is simply connected. If p and q are any two points then there is path between them and so there is at least one homotopy class of path from p to q . Suppose that γ_0 and γ_1 are two paths from p to q . Let

$$\gamma = \gamma_0 \cdot \gamma_1^{-1}.$$

Then γ is a loop based at p . As X is simply connected, γ is homotopic to the identity. But then γ_0 is homotopic to γ_1 .

2. As Y has the discrete topology,

$$X \times Y = \coprod_{y \in Y} X \times \{y\}$$

is a decomposition of $p^{-1}(X)$ into a disjoint union of open sets, all homeomorphic to X under the projection p . It follows that X is evenly covered, so that p is a covering map.

3. Pick a point $z \in Z$. We have to find an open neighbourhood of z that is evenly covered by p . As r is a covering map there is an open neighbourhood W of z which is evenly covered by r . It follows that

$$p^{-1}(W) = \coprod_{i=1}^k V_i,$$

where V_i are open subsets of Y , that are homeomorphic to W via r . Let $y_1, y_2, \dots, y_k$ be the points of Y that map to z . For each y_i , pick an open neighbourhood T_i of y_i evenly covered by q . Then $T_i \cap V_i$ is an open neighbourhood of Y_i evenly covered by q . So there is no harm in assuming that $T_i \subset V_i$. Let $W_i \subset W$ be the image of T_i . Then W_i is an open subset of W homeomorphic to T_i . Let

$$W_0 = \bigcap_{i=1}^k W_i.$$

Then W_0 is open, as it is the intersection of finitely many open subsets of Z . Replacing W by W_0 we may assume that each V_i is evenly covered

by q ,

$$q^{-1}(V_i) = \coprod_{j \in \Lambda_i} U_{ij}.$$

In this case

$$p^{-1}(W) = \coprod_{j \in \bigcup_{i=1}^k \Lambda_i} U_{ij}.$$

By assumption each U_{ij} is homeomorphic to V_i and each V_i is homeomorphic to W . Thus U_{ij} is homeomorphic to W . But then W is evenly covered by p and p is a covering map.

4. (i) Let

$$U_{x>0, y>0} = \{x + iy \in S^1 \mid x > 0, y > 0\}.$$

Then

$$p^{-1}(U_{y>0}) = U_{x>0, y>0} \cup U_{x<0, y<0}.$$

This is clearly a disjoint union. If we restrict p to the $U_{x>0, y>0}$ then the resulting map is a bijection, so that it is a homeomorphism. Similarly $U_{x<0, y<0}$ is homeomorphic to $U_{y>0}$. Thus $U_{y>0}$ is evenly covered by p . Similarly $U_{y<0}$, $U_{x>0}$ and $U_{x<0}$ are all evenly covered. Thus p is a covering map.

(ii) Consider the loop

$$\gamma: I \longrightarrow S^1 \quad \text{given by} \quad t \longrightarrow e^{2\pi it}.$$

The homotopy class of this loop generates the fundamental group. The composition

$$p \circ \gamma: I \longrightarrow S^1 \quad \text{is given by} \quad t \longrightarrow e^{4\pi it}.$$

Then

$$p \circ \gamma = \gamma \cdot \gamma,$$

so that

$$p_*: \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$$

is just multiplication by 2.

(iii) Let

$$U_{(\alpha, \beta)} = \{e^{i\theta} \in S^1 \mid \theta \in (\alpha, \beta)\} \quad \text{and} \quad U_i = U_{(2i\pi/n, (2i+1)\pi/n)}.$$

Then

$$p^{-1}(U_{y>0}) = p^{-1}(U_{(0, \pi)}) = \coprod_{i=0}^{n-1} U_{(2i\pi/n, (2i+1)\pi/n)} = \coprod_{i=0}^{n-1} U_i.$$

This is clearly a disjoint union. If we restrict p to the U_i then the resulting map is a bijection, so that it is a homeomorphism. Thus $U_{y>0}$ is evenly covered.

Similarly $U_{y<0}$, $U_{x>0}$ and $U_{x<0}$ are all evenly covered. Thus p is a covering map.

(iv) The composition of γ and p is

$$p \circ \gamma: I \longrightarrow S^1 \quad \text{is given by} \quad t \longrightarrow e^{2n\pi it}.$$

This is simply γ concatenated with itself n times, so that

$$p_*: \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$$

is just multiplication by n .

5. (i) The annulus is homeomorphic to $S^1 \times I$. As I is contractible, it follows that $S^1 \times I$ is homotopic to S^1 . Thus the annulus and the circle are homotopy equivalent. As the fundamental group of the circle is isomorphic to $\mathbb{Z}$, it follows that the fundamental group of the annulus is isomorphic to $\mathbb{Z}$.

(ii) We know that the map

$$p: \mathbb{R}^2 \longrightarrow S^1 \times S^1$$

given by

$$p(s, t) = (e^{2\pi is}, e^{2\pi it})$$

is a covering map. $\mathbb{R}^2$ is contractible so that it is certainly simply connected. Thus p is the universal cover. The fibre over the origin is the grid of points with integer points

$$\mathbb{Z}^2 \subset \mathbb{R}^2.$$

Thus the fundamental group of the torus is in bijection with $\mathbb{Z}^2$.

(iii) We already saw that the fundamental group of the sphere S^2 is trivial. By simplicial approximation every continuous map from S^1 to the S^2 is homotopic to a continuous map which avoids a point. But we can homotopy any such map to a point.

(iv) There is a covering map

$$p: S^2 \longrightarrow \mathbb{RP}^2.$$

S^2 is simply connected, so that p is the universal cover. p is a double cover, so that the order of the fundamental group is two. Thus the fundamental group is isomorphic to $\mathbb{Z}_2$.