

MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. Suppose to the contrary that there is no point y such that $f(y) = -y$. Define a function

$$F: S^1 \times I \longrightarrow S^1$$

by the rule

$$F(x, t) = \frac{v}{\|v\|} \quad \text{where} \quad v = t(x - f(x)) + f(x).$$

Note that

$$t(x - f(x)) + f(x) \quad \text{for} \quad t \in [0, 1]$$

describes the line segment from $f(x)$ to x . If $f(x) \neq -x$ then this line segment does not pass through the origin and v is never zero and so F is a continuous function. We have

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = x.$$

Thus F is a homotopy between $f(x)$ and the identity.

Now consider the function

$$g: S^1 \longrightarrow S^1 \quad \text{given by} \quad g(x) = -f(x).$$

Then g is a continuous function. If there is no X such that $f(x) = x$ then there is no y so that $g(y) = y$. But then by what we just proved, g is homotopic to the identity.

Since f is the composition of g and the antipodal map $x \longrightarrow -x$, it follows that f is homotopic to the antipodal map $x \longrightarrow -x$. But we already saw in Homework 7, Question 2, that this is homotopic to the identity.

Thus if f is not homotopic to the identity then there is a point y such that $f(y) = -y$.

2. (i) A convex set is clearly star convex about any point.

The union of the x and y -axis in $\mathbb{R}^2$ is star convex (about the origin) but not convex.

(ii) We show that a star convex set is contractible. Define a function

$$F: A \times I \longrightarrow A \quad \text{by the rule} \quad F(x, t) = (1 - t)(x - a_0) + a_0.$$

Note that

$$(1 - t)(x - a_0) + a_0$$

is the line segment between x_0 and x . By assumption this lies in A . It is clear that F is continuous. We have

$$F(x, 0) = (x - a_0) + a_0 = x \quad \text{and} \quad F(x, 1) = a_0.$$

Thus F is a homotopy from the identity to the constant function to a_0 . It follows that A is contractible. But then A is simply connected.

3. Let

$$[\gamma] \in \pi_1(A, a_0).$$

Let $i: A \rightarrow X$ be the natural inclusion. Consider

$$i_*[\gamma] \in \pi_1(X, a_0).$$

We have

$$\begin{aligned} r_*(i_*[\gamma]) &= (r \circ i)_*[\gamma] \\ &= (\text{id}_A)_*[\gamma] \\ &= [\gamma]. \end{aligned}$$

Thus r_* is surjective.

4. The two projection maps

$$p: X \times Y \rightarrow X \quad \text{and} \quad q: X \times Y \rightarrow Y$$

are continuous and send the base point to the base point. Thus we get two group homomorphisms

$$p_*: \pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(X, x_0) \quad \text{and} \quad q_*: \pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(Y, y_0).$$

By the universal property of the product (of groups) there is then an induced group homomorphism

$$\phi: \pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

We first show that ϕ is surjective. There are two ways to proceed.

For the first, suppose that $(\alpha, \beta) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$. Note that there is an obvious loop in $X \times Y$,

$$\gamma: I \rightarrow X \times Y \quad \text{given by} \quad \delta(t) = (\alpha(t), y_0).$$

γ is a loop that maps to α ,

$$p_*\gamma = \alpha.$$

Similarly we may find a loop δ in $X \times Y$ such that

$$q_*\delta = \beta.$$

It follows that ϕ is surjective.

For the second, observe that p is retraction of the natural inclusion map

$$i: X \rightarrow X \times Y \quad \text{given by} \quad i(x) = (x, y_0).$$

Thus p_* is surjective. Similarly q_* is surjective and it follows that easily that ϕ is surjective.

Suppose that $[\gamma]$ belongs to the kernel of ϕ . Then

$$\alpha = p \circ \gamma \quad \text{and} \quad \beta = q \circ \gamma$$

are loops in X and Y that are homotopic to the constant maps. Let A and B be the corresponding homotopies. Define a function

$$C: I^2 \longrightarrow X \times Y$$

by the rule

$$C(x, y, t) = (A(s, t), B(s, t)).$$

Then C is continuous, as its components are continuous. We have

$$\begin{aligned} C(s, 0) &= (A(s, 0), B(s, 0)) \\ &= (\alpha(s), \beta(s)) \\ &= \gamma(s) \end{aligned}$$

and

$$\begin{aligned} C(s, 1) &= (A(s, 1), B(s, 1)) \\ &= (\alpha(1), \beta(1)) \\ &= (x_0, y_0). \end{aligned}$$

Thus C is a homotopy from γ to the identity, so that γ is homotopic to the identity. Thus the kernel of ϕ is trivial and ϕ is an isomorphism.

5. (i) This binary operation is simply pointwise multiplication. Associativity follows easily, as $\star$ is associative. The constant function c_{x_0} plays the role of the identity. If γ is a loop then the loop δ which to every point of I assigns the inverse in the group G is then the inverse loop. Thus $\Omega(G, x)$ is a group with this rule of multiplication.

(ii) We have to check that if $\alpha_0 \sim \alpha_1$ and $\beta_0 \sim \beta_1$ are four loops then

$$\gamma_0 = \alpha_0 \diamond \beta_0 \sim \alpha_1 \diamond \beta_1 = \gamma_1.$$

By assumption we are given homotopies A and B between α_0 and α_1 and β_0 and β_1 . Define a function

$$C: I^2 \longrightarrow G$$

by the rule

$$C(s, t) = A(s, t) \star B(s, t).$$

Then C is continuous as it is the composition of the function

$$(s, t) \longrightarrow (A(s, t), B(s, t))$$

which is continuous as its components are continuous and multiplication in the group G , with is by assumption continuous. We have

$$\begin{aligned} C(s, 0) &= A(s, 0) \star B(s, 0) \\ &= \alpha_0(s) \star \beta_0(s) \\ &= (\alpha_0 \diamond \beta_0)(s) \\ &= \gamma_0(s), \end{aligned}$$

and

$$\begin{aligned} C(s, 1) &= A(s, 1) \star B(s, 1) \\ &= \alpha_1(s) \star \beta_1(s) \\ &= (\alpha_1 \diamond \beta_1)(s) \\ &= \gamma_1(s). \end{aligned}$$

Thus C is a homotopy between γ_0 and γ_1 , so that $\gamma_0 \sim \gamma_1$.

(iii) We compute $(f \cdot c_{x_0}) \diamond (c_{x_0} \cdot g)$ in two different ways. If we compute one way then we get

$$(f \cdot c_{x_0}) \diamond (c_{x_0} \cdot g) = f \diamond g$$

But if distribute one multiplication over the other we also get

$$\begin{aligned} (f \cdot c_{x_0}) \diamond (c_{x_0} \cdot g) &= (f \diamond c_{x_0}) \cdot (c_{x_0} \diamond g) \\ &= f \cdot g. \end{aligned}$$

Note that in the first line the c_{x_0} move from one bracket to the other. Thus

$$f \diamond g = f \cdot g.$$

(iv) Note that

$$(f \diamond g)^{-1} = g^{-1} \diamond f^{-1}.$$

Indeed, this just reflects how to compute inverses in G . But

$$(f \cdot g)^{-1} = f^{-1} \cdot g^{-1}.$$

Indeed, this is simply because composition of paths is concatenation, so we don't change the order. Since the two multiplications come out the same, it follows that

$$g^{-1} \cdot f^{-1} = f^{-1} \cdot g^{-1}.$$

Since this is true for any f and g we must have

$$f \cdot g = g \cdot f.$$

But then $\pi_1(G, x_0)$ is abelian.

Challenge Problems: (Just for fun)

6. We have

$$\begin{aligned}
1_\star &= 1_\star \star 1_\star \\
&= (1_\diamond \diamond 1_\star) \star (1_\star \diamond 1_\diamond) \\
&= (1_\diamond \star 1_\star) \diamond (1_\star \star 1_\diamond) \\
&= 1_\diamond \diamond 1_\diamond \\
&= 1_\diamond.
\end{aligned}$$

From now on we can drop the subscript,

$$1_\star = 1 = 1_\diamond.$$

Now suppose that a and $b \in A$ are two elements of A . We have

$$\begin{aligned}
a \star b &= (1 \diamond a) \star (b \diamond 1) \\
&= (1 \star b) \diamond (a \star 1) \\
&= b \diamond a \\
&= (b \star 1) \diamond (1 \star a) \\
&= (b \diamond 1) \star (1 \diamond a) \\
&= b \star a.
\end{aligned}$$

Thus $\star = \diamond$ and both operations are commutative.

Finally suppose that a, b and $c \in A$ are three elements of A . We have

$$\begin{aligned}
(a \star b) \star c &= (a \star b) \star (1 \star c) \\
&= (a \star 1) \star (b \star c) \\
&= a \star (b \star c).
\end{aligned}$$

This is associativity.