

MODEL ANSWERS TO THE SEVENTH HOMEWORK

1. (i) If f has no fixed points then there is always a unique line connecting $f(x)$ to x . Continuing in this direction this line meets the unit sphere at a point $r(x)$.

In this way we get a function

$$r: X \longrightarrow S^{n-1}.$$

Note that r is the composition of the function

$$X \longrightarrow X \times X \quad \text{given by} \quad x \longrightarrow (x, f(x))$$

and the function l . Thus r is continuous.

If x belongs to the unit sphere then the line connecting $f(x)$ to x meets the unit sphere at the point x . Thus $r \circ i = \text{id}_{S^{n-1}}$ and so r is a retraction.

(ii) The two spaces X and A are triangulable. There are three continuous maps, i , r and $\text{id}_{S^{n-1}}$. Each map gives us a pushforward on homology.

$$i_*: H_m(S^{n-1}) \longrightarrow H_m(X) \quad r_*: H_m(X) \longrightarrow H_m(S^{n-1}) \quad \text{and} \quad \text{id}_{S^{n-1}*}: H_m(S^{n-1}) \longrightarrow H_m(S^{n-1})$$

We concentrate on the case $m = n - 1$. Now the first map is the zero map, as $H_{n-1}(X) = 0$. The last map is the identity. On other hand,

$$\text{id}_{S^{n-1}*} = r_* \circ i_*,$$

at least up to isomorphism. But the LHS is the identity map $\mathbb{Z} \longrightarrow \mathbb{Z}$ and the RHS is the zero map.

As this is not possible there is no retraction. But then every map f must have a fixed point.

2. The antipodal map on S^1 represents rotation through 180° . Thus the obvious homotopy is simply to continuously change the angle of rotation. Now rotation through θ is represented by the matrix

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

The corresponding linear map is

$$(x, y) \longrightarrow (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta).$$

Thus we define

$$F: S^1 \times I \longrightarrow S^1 \quad \text{by the rule} \quad (x, y) \longrightarrow (x \cos(\pi - t) - y \sin(\pi - t), x \sin(\pi - t) + y \cos(\pi - t)).$$

This is obviously a continuous map and the image lands in S^1 as we are simply rotating the coordinates. We have

$$F(x, y, 0) = (-x, -y) = a(x, y) \quad \text{and} \quad F(x, y, 1) = (x, y) = \text{id}_{S^1}(x, y).$$

More generally, if n is odd, then we can pair of the coordinates of R^{n+1} , and apply the map on each of coordinates.

$$(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, t) \longrightarrow (x_1 \cos(\pi-t) - y_1 \sin(\pi-t), x_1 \sin(\pi-t) + y_1 \cos(\pi-t), \dots).$$

This gives us a continuous function

$$F: S^n \times I \longrightarrow S^n$$

which is a homotopy from a to the identity.

3. We compute

$$\begin{aligned} h \circ f &= \text{id}_X \circ (h \circ f) \\ &\sim (g \circ f) \circ (h \circ f) \\ &= g \circ (f \circ h) \circ f \\ &\sim g \circ (\text{id}_Y \circ f) \\ &= g \circ f \\ &\sim \text{id}_X. \end{aligned}$$

Thus h is a homotopy inverse of f . Similarly one can check that $f \circ g$ is homotopic to the identity, by a similar argument. It follows that g is a homotopy inverse of f .

But then f , g and h are homotopy equivalences.

4. Let X be a topological space, let A be a subspace and let $i: A \longrightarrow X$ be the natural inclusion. Suppose that A is a retraction of X , so that there is a continuous function $r: X \longrightarrow A$ such that $r \circ i = \text{id}_A$.

Suppose that X is contractible. Then there is a continuous map

$$F: X \times I \longrightarrow X$$

which is a homotopy from the identity to a constant map,

$$F(x, 0) = x \quad \text{and} \quad F(x, 1) = x_0,$$

a fixed point $x_0 \in X$. Pick a point $a_0 \in A$. Then there is no harm in assuming that $x_0 = a_0$.

Let

$$G = r \circ F \circ i: A \times A \longrightarrow A$$

be the composition, on the left by r and on the right by i . Then G is continuous, as it is the composition of continuous functions and

$$\begin{aligned} G(a, 0) &= r \circ F(a, 0) \\ &= r(a) \\ &= a. \end{aligned}$$

On the other hand,

$$\begin{aligned} G(a, 1) &= r \circ F(a, 1) \\ &= r(a_0) \\ &= a_0. \end{aligned}$$

Thus A is contractible.

5. (i) Recall that if we delete a point p from S^n then we get something homeomorphic to $\mathbb{R}^n$.

Let

$$g: S^n \setminus \{p\} \longrightarrow \mathbb{R}^n$$

be a homeomorphism. Let $h: \mathbb{R}^n \longrightarrow \{p\}$ be a constant map to a point. We have

$$\begin{aligned} f &= g^{-1} \circ g \circ f \\ &= g^{-1} \circ \text{id}_{\mathbb{R}^n} \circ g \circ f \\ &\sim g^{-1} \circ h \circ g \circ f. \end{aligned}$$

The last map is a constant map to a point. Thus f is homotopic to a constant function.

(ii) Pick a triangulations $\mathcal{K}$ and $\mathcal{L}$ of S^m and S^n . Then we may assume that we have a continuous map

$$h: |\mathcal{K}| \longrightarrow |\mathcal{L}|$$

By the Simplicial Approximation Theorem, if we replace $\mathcal{K}$ by a subcomplex then we may assume that h has a simplicial approximation

$$f: \mathcal{K} \longrightarrow \mathcal{L}.$$

By Homework 5, Question 1, f and h are homotopic. Let σ be a simplex in $\mathcal{K}$. Then its image is a simplex τ in $\mathcal{L}$. As the dimension of σ is at most m , it follows that τ has dimension at most m .

Pick a simplex μ in $\mathcal{L}$ whose dimension is n . Then no point in the interior is in the image of f .

Thus f is homotopic to a constant map, by (i).