

MODEL ANSWERS TO THE SIXTH HOMEWORK

1. By (12.2) there is an integer s such that

$$\mu(\mathcal{L}^{(s)}) < \frac{\epsilon}{2}.$$

By the simplicial approximation theorem there is then an integer r and a simplicial approximation f from $\mathcal{K}^{(r)}$ to $\mathcal{L}^{(s)}$ of the continuous function h .

Suppose that $x \in |\mathcal{K}|$. Pick a vertex v of $\mathcal{K}$ such that

$$x \in \text{St}(v).$$

Then

$$h(x) \in h(\text{St}(v)) \subset \text{St}(f(v))$$

It follows that

$$\|h(x) - f(v)\| < \frac{\epsilon}{2}.$$

By assumption x belongs to the interior of a simplex σ and v is a vertex of σ . As f is simplicial $f(x)$ belongs to the interior of simplex τ containing v . Thus

$$\|f(x) - f(v)\| < \frac{\epsilon}{2}.$$

It follows that

$$\begin{aligned} \|f(x) - h(x)\| &\leq \|f(x) - f(v)\| + \|f(v) - h(x)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

2. The objects are chain complexes $C_\bullet$ and the morphisms are chain maps. We check the axioms for a category.

Given three chain complexes $C_\bullet$, $D_\bullet$ and $E_\bullet$ and two chain maps $f_\bullet$ and $g_\bullet$ from $C_\bullet$ to $D_\bullet$ and $D_\bullet$ to $E_\bullet$, we just compose the corresponding group homomorphisms to get $h_\bullet$ from $C_\bullet$ to $E_\bullet$.

$$h_m = g_m \circ f_m: C_m \longrightarrow E_m$$

We check that we do have a chain map

$$\begin{aligned}
\delta_m \circ h_m &= \delta_m \circ (g_m \circ f_m) \\
&= (\delta_m \circ g_m) \circ f_m \\
&= (g_m \circ \delta_m) \circ f_m \\
&= g_m \circ (\delta_m \circ f_m) \\
&= g_m \circ (f_m \circ \delta_m) \\
&= (g_m \circ f_m) \circ \delta_m \\
&= h_m \circ \delta_m.
\end{aligned}$$

Thus $h_\bullet$ is a chain map.

Composition is associative, as composition of functions is associative.

The map which is the identity on C_m , for all m , is a chain map and it obviously plays the role of the identity.

Thus we have a category.

3. A sequence of group homomorphisms

$$P_m: C_m \longrightarrow D_{m+1}$$

such that

$$\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m = f - g.$$

(i) Suppose that f is a chain map from $C_\bullet$ to $D_\bullet$. We define

$$P_m: C_m \longrightarrow D_{m+1}$$

to be the zero map for all m .

Since $f - f$ is the zero map, the equation

$$\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m = f - f$$

holds, since both sides are the zero map. Thus $f \sim f$ and $\sim$ is reflexive.

Now suppose that f and g are two chain maps from $C_\bullet$ to $D_\bullet$ and $f \sim g$. It follows that we can find a sequence of group homomorphisms

$$P_m: C_m \longrightarrow D_{m+1}$$

such that

$$\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m = f - g.$$

Define a sequence of group homomorphisms

$$Q_m: C_m \longrightarrow D_{m+1} \quad \text{by the rule} \quad Q_m(\alpha) = -P_m(\alpha).$$

We have

$$\begin{aligned}
(\partial_{m+1} \circ Q_m + Q_{m-1} \circ \partial_m)(\alpha) &= -(\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m)(\alpha) \\
&= -f(\alpha) + g(\alpha) \\
&= (g - f)(\alpha).
\end{aligned}$$

Thus

$$\partial_{m+1} \circ Q_m + Q_{m-1} \circ \partial_m = g - f$$

so that $g \sim f$. Thus $\sim$ is symmetric.

Now suppose that f , g and h are three chain maps from $C_\bullet$ to $D_\bullet$ and $f \sim g$, $g \sim h$. It follows that we can find two sequences of group homomorphisms

$$P_m: C_m \longrightarrow D_{m+1}$$

such that

$$\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m = f - g$$

and

$$Q_m: C_m \longrightarrow D_{m+1}$$

such that

$$\partial_{m+1} \circ Q_m + Q_{m-1} \circ \partial_m = g - h.$$

Let

$$R_m: C_m \longrightarrow D_{m+1}$$

be the sequence of group homomorphisms

$$R_m(\alpha) = P_m(\alpha) + Q_m(\alpha).$$

If we compose on the left and the right and use linearity of ∂ then we get

$$\begin{aligned} \partial_{m+1} \circ R_m &= \partial_{m+1} \circ P_m + \partial_{m+1} \circ Q_m \\ R_{m-1} \circ \delta_m &= P_{m-1} \circ \delta_m + Q_{m-1} \circ \delta_m. \end{aligned}$$

We compute

$$\begin{aligned} (\partial_{m+1} \circ R_m + R_{m-1} \circ \partial_m)(\alpha) &= (\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m)(\alpha) + (\partial_{m+1} \circ Q_m + Q_{m-1} \circ \partial_m)(\alpha) \\ &= f(\alpha) - g(\alpha) + g(\alpha) - h(\alpha) \\ &= (f - h)\alpha. \end{aligned}$$

Thus $f \sim h$ and $\sim$ is transitive. Hence $\sim$ is an equivalence relation.

(ii) Suppose that we have chain maps $f_1 \circ f_2$ from $C_\bullet$ to $D_\bullet$ and chain maps $g_1 \sim g_2$ from $D_\bullet$ to $E_\bullet$.

We have to show that the two chain maps $g_1 \circ f_1$ and $g_2 \circ f_2$ from $C_\bullet$ to $E_\bullet$ are chain homotopic, $g_1 \circ f_1 \sim g_2 \circ f_2$.

We do this in two stages. We show that if $g_1 \sim g_2$ then $g_1 \circ f \sim g_2 \circ f$ and if $f_1 \sim f_2$ then $g \circ f_1 \sim g \circ f_2$.

By assumption we may find a sequence of group homomorphisms

$$P_m: C_m \longrightarrow D_{m+1}$$

such that

$$\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m = f_1 - f_2.$$

We define a sequence of group homomorphisms

$$Q_m = g_{m+1} \circ P_m : C_m \longrightarrow E_{m+1}$$

as the composition. We compute

$$\begin{aligned} (\partial_{m+1} \circ Q_m + Q_{m-1} \circ \partial_m)(\alpha) &= (\partial_{m+1} \circ Q_m)(\alpha) + (g_{m-2} \circ P_{m-1}) \circ \partial_m(\alpha) \\ &= (\partial_{m+1} \circ (g_{m+1} \circ P_m))(\alpha) + g_{m-2} \circ (P_{m-1} \circ \partial_m)(\alpha) \\ &= ((\partial_{m+1} \circ g_{m+1}) \circ P_m)(\alpha) + g_{m-2} \circ (P_{m-1} \circ \partial_m)(\alpha) \\ &= ((g_{m+1} \circ \partial_{m+1}) \circ P_m)(\alpha) + g_{m-2} \circ (P_{m-1} \circ \partial_m)(\alpha) \\ &= (g_{m+1} \circ (\partial_{m+1} \circ P_m))(\alpha) + g_{m-2} \circ (P_{m-1} \circ \partial_m)(\alpha) \\ &= g_{m+1} \circ (\partial_{m+1} \circ P_m)(\alpha) + P_{m-1} \circ (\partial_m)(\alpha) \\ &= g \circ (f_1 - f_2)(\alpha) \\ &= (g \circ f_1 - g \circ f_2)(\alpha) \\ &= ((g \circ f_1) - (g \circ f_2))(\alpha). \end{aligned}$$

Thus $g \circ f_1$ is chain homotopic to $g \circ f_2$.

Now suppose that $g_1 \circ g_2$. By assumption we may find a sequence of group homomorphisms

$$P_m : D_m \longrightarrow E_{m+1}$$

such that

$$\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m = g_1 - g_2.$$

We define a sequence of group homomorphisms

$$Q_m = P_m \circ f_m : C_m \longrightarrow E_{m+1}$$

as the composition. A similar calculation to the one above shows that Q_m is a chain homotopy from $g_1 \circ f$ to $g_2 \circ f$.

(iii) This is almost line by line the same proof as the case when there are simplicial complexes underlying our chain complexes. Suppose we are given $\alpha \in Z_m(C_\bullet)$. We have

$$\begin{aligned} f(\alpha) - g(\alpha) &= (f - g)(\alpha) \\ &= (\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m)(\alpha) \\ &= (\partial_{m+1} \circ P_m)(\alpha) + (P_{m-1} \circ \partial_m)(\alpha) \\ &= (\partial_{m+1}(P_m(\alpha))) + (P_{m-1}(\partial_m \alpha)) \\ &= \partial_{m+1} \beta, \end{aligned}$$

where $\beta = P_{m+1}(\alpha) \in D_{m+1}$. Thus $f(\alpha)$ and $g(\alpha)$ are homologous. It follows that they are equal in homology, so that they induce the same map in homology.

(iv) If $f \circ g$ is chain homotopic to the identity then

$$f_* \circ g_* = (f \circ g)_*$$

is equal to the identity on homology, by (iii). Similarly if $g \circ f$ is chain homotopic to the identity then $g \circ f$ is equal to the identity on homology.

Thus f_* and g_* are inverses to each other in homology.

(v) By assumption we may find chain maps

$$f': D_\bullet \longrightarrow C_\bullet \quad \text{and} \quad g': E_\bullet \longrightarrow D_\bullet$$

such that $f \circ f'$, $f' \circ f$, $g \circ g'$ and $g' \circ g$ are all chain homotopic to the identity. Let $h = g \circ f$ and $h' = f' \circ g'$. We check that $h \circ h'$ and $h' \circ h$ are chain homotopic to the identity. We have

$$\begin{aligned} h \circ h' &= (g \circ f) \circ (f' \circ g') \\ &= g \circ ((f \circ f') \circ g') \\ &\sim g \circ (\text{id} \circ g') \\ &= g \circ g' \\ &\sim \text{id}. \end{aligned}$$

A similar argument, the other way around, shows that $h' \circ h$ is chain homotopic to the identity.

4. We just need to find a triangulation of each space and compute its homology. Fortunately we already did this. Let σ be an n -simplex. Let $\mathcal{K}$ be the simplicial complex given by σ and all its faces. $\mathcal{K}$ gives a triangulation of the closed unit ball B . Hence

$$H_m(B) \simeq H_m(\mathcal{K}) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\mathcal{L}$ be the complex you get by deleting σ . We shift dimensions and assume that σ is an $(n+1)$ -simplex. Then $\mathcal{L}$ is a triangulation of S^n . Hence

$$H_m(S^n) \simeq H_m(\mathcal{L}) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$$

In the other cases we gave explicit triangulations of a square that when we quotient out give triangulations of the given surface. The results are: for the annulus

$$H_m(S^1 \times I) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \text{ or } 1 \\ 0 & \text{otherwise;} \end{cases}$$

for the torus

$$H_m(S^1 \times S^1) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \text{ or } 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } m = 1 \\ 0 & \text{otherwise;} \end{cases}$$

for the Klein bottle

$$H_m(K^2) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } m = 1 \\ 0 & \text{otherwise;} \end{cases}$$

and for the real projective plane

$$H_m(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \\ \mathbb{Z}_2 & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Challenge Problems: (Just for fun)

5. (i) Start with an element $\beta_3 \in B_3$. It maps to an element β_4 in B_4 . As h_4 is the surjective, this lifts to an element α_4 in A_4 . Let α_5 be the image of α_4 in A_5 . Let β_5 be the image of α_5 in B_5 . Since we have a commutative diagram this is the same as the image of β_4 . Since β_4 is the image of β_3 it follows that $\beta_5 = 0$. As h_5 is injective it follows that α_5 is zero. As the top row is exact it follows that we may lift α_4 to α_3 in A_3 . Let β'_3 be the image of α_3 in B_3 . As the diagram commutes the image of β'_3 in B_4 is β_4 . Thus the image of $\delta_3 = \beta_3 - \beta'_3$ in B_4 is zero. But then we may find β_2 in B_2 mapping to δ_3 . As h_2 is surjective it follows that we may α_2 in A_2 mapping to β_2 . Let γ_3 be the image of α_2 in A_3 .

The image of γ_3 in B_3 is δ_3 , as the diagram commutes. Consider $\alpha = \alpha_3 + \gamma_3$. Its image in B_3 is

$$\begin{aligned} \beta'_3 + \delta_3 &= \beta'_3 + \beta_3 - \beta'_3 \\ &= \beta_3. \end{aligned}$$

Thus h_3 is surjective.

(ii)