

MODEL ANSWERS TO THE FIFTH HOMEWORK

1. (i) The function

$$\alpha: X \longrightarrow \mathbb{R}^{2n+1} \quad \text{given by} \quad x \longrightarrow (f(x), g(x), t)$$

is continuous, as the three components, f , g and the identity $\text{id}_{\mathbb{R}}$ are continuous. The function

$$\beta: \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^n \quad \text{given by} \quad (x, y, t) \longrightarrow (1-t)x + ty$$

is continuous, since it is a polynomial in x , y and t . The function

$$F = \beta \circ \alpha: X \times I \longrightarrow \mathbb{R}^n \quad \text{given by} \quad F(x, t) = (1-t)f(x) + tg(x)$$

is continuous, as it is the composition of α and β . On the other hand,

$$\begin{aligned} F(x, 0) &= (1-0) \cdot f(x) + 0 \cdot g(x) \\ &= f(x), \end{aligned}$$

and

$$\begin{aligned} F(x, 1) &= (1-1) \cdot f(x) + 1 \cdot g(x) \\ &= g(x). \end{aligned}$$

Thus F is a homotopy from f to g .

(ii) We simply use the map F (with h replacing g). We have only to show that the image of F always lands in $|\mathcal{L}|$. Pick $x \in |\mathcal{K}|$. Then x belongs to the interior of some simplex σ . Suppose that $h(x)$ belongs to the interior of τ . We saw in the proof of (10.2) that the images of the vertices of σ are vertices of τ . Thus

$$f(\sigma) \subset \tau \quad \text{and} \quad h(x) \in \tau.$$

The map

$$I \longrightarrow \mathbb{R}^n \quad \text{given by} \quad t \longrightarrow (1-t)f(x) + th(x)$$

describes the straight line segment from $f(x)$ to $h(x)$. Since $f(x)$ and $h(x)$ belong to τ the whole line segment belongs to τ . But then the image of F lands in $|\mathcal{L}|$ and F is a homotopy from f to h .

2. If $m = 0$ then σ is a single vertex V_0 . In this case $\mathcal{L}$ is also just the single vertex v_0 .

If $m = 1$ then $\sigma = \langle v_0, v_1 \rangle$. $\mathcal{L}$ has five simplices, v_0 , v_1 , $\hat{\sigma}$, $\langle v_0, \hat{\sigma} \rangle$ and $\langle v_1, \hat{\sigma} \rangle$.

If $m = 2$ then there are $7 = 2^3 - 1$ vertices, 6 2-simplices and 12 edges. This gives 25 simplices.

If $m = 3$ then there are $15 = 2^4 - 1$ vertices, the number of subsets of the integers from 0 to 3, excluding the empty set. There are $4 \times 6 = 24$ 3-simplices (the cones over the 2-simplices in the four faces). 1-simplices are in correspondence with a pair of nested subsets of the four integers from 0 to 3. This means there are

$$4 \cdot \left(\binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right) + \binom{4}{2} \cdot \left(\binom{2}{1} + \binom{2}{2} \right) + \binom{4}{3}$$

that is, there are 50 edges.

2-simplices come in two types. There are the cones over the 1-simplices in the faces. There are 14 edges not contained in the faces. It follows that there are $36 = 50 - 14$ edges contained in the faces. Thus there are 36 2-simplices which are cones over 1-simplices in the faces.

There are also the 2-simplices contained in the faces. There are four faces and they each contain six 2-simplices. The intersection of two faces is a 1-simplex and so we have not over counted. Thus there are 24 2-simplices in the faces.

It follows that there are 60 2-simplices. In total there are then

$$15 + 50 + 60 + 24 = 149.$$

3. It is enough to do (ii). Let $\mathcal{M}_0$ be the union of all simplices τ that don't contain x that are faces of simplices σ that do contain x . Let $\mathcal{M}$ be the cone over $\mathcal{M}_0$.

Now let

$$\mathcal{L} = \mathcal{K} \setminus \mathcal{M}_0 \cup \mathcal{M}.$$

We check that $\mathcal{L}$ is a subdivision of $\mathcal{K}$.

We first check that $\mathcal{L}$ is a simplicial complex. Suppose that μ is a simplex in $\mathcal{L}$ and that ν is a face of μ . If μ belongs to $\mathcal{M}$ then ν belongs to $\mathcal{M}$ and so ν belongs to $\mathcal{L}$. If μ belongs to $\mathcal{K} \setminus \mathcal{M}_0$ then μ is not a face of any simplex that contains x . Then ν belongs to $\mathcal{K}$ and ν does not contain x , so that $\nu \in \mathcal{K} \setminus \mathcal{M}_0$. Thus $\nu \in \mathcal{L}$.

Now suppose that μ and $\nu \in \mathcal{L}$ and $\rho = \mu \cap \nu$ is non-empty. If μ and ν both belong to $\mathcal{M}$ then ρ is a face of both μ and ν . If μ and ν both belong to $\mathcal{K} \setminus \mathcal{M}_0$ then ρ is also a face of both μ and ν . Suppose that $\mu \in \mathcal{M}$ and $\nu \in \mathcal{K} \setminus \mathcal{M}_0$. Then ν is not a face of a simplex that contains x . Thus μ and ν don't intersect.

It follows that $\mathcal{L}$ is a simplicial complex. It is clear that the support of $\mathcal{M}$ is equal to the support of $\mathcal{M}_0$. Thus the support of $\mathcal{K}$ and $\mathcal{L}$ coincide. Finally every simplex in $\mathcal{L}$ belongs to a simplex in $\mathcal{K}$.

Thus $\mathcal{L}$ is a subdivision of $\mathcal{K}$. It is clear that we only added the vertex x .

4. (i) Let $\mathcal{U}$ be an open cover. Let $\mathcal{V}$ be the set of all singleton subsets of X . It is clear that $\mathcal{V}$ is an open cover of X . If $V \in \mathcal{V}$ then we may find $U \in \mathcal{U}$ such that $V \subset U$ as $\mathcal{U}$ covers X . Thus $\mathcal{V}$ is a refinement of $\mathcal{U}$.

It is clear that every point is contained in exactly one element of $\mathcal{V}$. Thus the covering dimension is zero.

(ii) Let $\mathcal{U}$ be an open cover. Note that a refinement of a refinement is a refinement. Passing to a refinement, we may therefore assume that the elements of $\mathcal{U}$ are intervals. As I is compact, $\mathcal{U}$ has a finite subcover $\mathcal{V}$. Note that a subcover is a refinement. Replacing $\mathcal{U}$ by $\mathcal{V}$ we may therefore assume that $\mathcal{U}$ is finite.

We may also assume that $\mathcal{U}$ is minimal with respect to inclusion. If $\mathcal{V}$ is a subcover of $\mathcal{U}$ then $\mathcal{U} = \mathcal{V}$.

Suppose that $r \in I$ and let $\mathcal{R}$ be the elements of $\mathcal{U}$ that contain r . Let $U_s \in \mathcal{R}$ be an element of $\mathcal{R}$ that contains the smallest elements of I (this makes sense as $\mathcal{R}$ is finite). Similarly let $U_b \in \mathcal{R}$ be an element of $\mathcal{R}$ that contains the biggest elements of I . Then every element of $\mathcal{R}$ belongs to the union $U_s \cup U_b$. This contradicts the fact that $\mathcal{U}$ is minimal.

Thus every element of I is contained in at most two elements of $\mathcal{U}$ and the covering dimension is 1.

(iii) Let $\mathcal{U}$ be an open cover of Y . If $U \in \mathcal{U}$ then there is an open set $U_1 \subset X$ such that $U_1 \cap Y = U$. If we set take all of these open sets and $X \setminus Y$ this gives us an open cover of X . By assumption there is an open cover of X , a refinement $\mathcal{V}_1$ of $\mathcal{U}_1$, such that every point of X is contained in at most $m + 1$ elements of $\mathcal{V}_1$.

Let

$$\mathcal{V} = \{ V_1 \cap Y \mid V_1 \in \mathcal{V}_1 \}.$$

Then $\mathcal{V}$ is an open cover of Y and a refinement of $\mathcal{U}$. It is clear that every element of Y is contained in at most $m + 1$ elements of $\mathcal{V}$.

(iv) Suppose we are given an open cover $\mathcal{U}$ of $|\mathcal{K}|$. Let δ be the Lebesgue number of the open cover. Pick $r > 0$

$$\mu(\mathcal{L}) < \delta \quad \text{where} \quad \mathcal{L} = \mathcal{K}^{(r)}.$$

Let $\mathcal{V}$ be the open cover of $|\mathcal{K}|$ given by the stars of $\mathcal{L}$

$$\mathcal{V} = \{ \text{St}(v, \mathcal{L}) \mid v \text{ a vertex of } \mathcal{L} \}.$$

Then if $U \in \mathcal{U}$ and $v \in V$ then

$$\text{St}(v, \mathcal{L}) \subset U.$$

Thus $\mathcal{V}$ is a refinement of $\mathcal{U}$. If $x \in |\mathcal{K}|$ then x belongs to the interior of a simplex σ of $\mathcal{L}$. In this case x belongs to $\text{St}(v, \mathcal{L})$ if and only if v is a vertex of σ . If σ is a p -simplex then σ has $p + 1 \leq n + 1$ vertices. Thus the covering dimension $m \leq n$.

5. We want to define a group homomorphism j and a chain homotopy P ,

$$j: C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{L}) \quad \text{and} \quad P_m: C_m(\mathcal{K}) \longrightarrow C_{m+1}(\mathcal{K}).$$

The first thing to do is to figure out happens on the subgroup $C_m(\mathcal{L}) \subset C_m(\mathcal{K})$. We just have to decide where to send a simplex $\rho \in \mathcal{L}$.

We want $j \circ i_*$ to be the identity. This forces j be the identity on the subgroup $C_m(\mathcal{L}) \subset C_m(\mathcal{K})$. In other words we put $j(\rho) = \rho$.

Note that roughly speaking P_m is the difference between $i_* \circ j$ and the identity. $i_* \circ j$ is close to the identity and so P_m is close to zero. In fact we choose P_m the zero map on $C_m(\mathcal{L}) \subset C_m(\mathcal{K})$. In other words we put $j(\rho) = 0$.

Now we have to figure out how to handle σ and τ . Note that since τ is a face of σ it follows that there is a basic connection between σ and τ ,

$$\partial_m \sigma = a\tau + \alpha,$$

where $a = \pm 1$ and α is not supported on τ . This dictates our choices. Possibly flipping the orientation of τ we might as well assume that $a = 1$. σ is an m -simplex and so we put $P_m(\sigma) = 0$. Similarly we put $j(\sigma) = 0$. The choice of where to send τ is more complicated. After some experimentation we realise we want to put $P_{m-1}(\tau) = \sigma$ and $j(\tau) = -\alpha$.

Putting all of this together we have

$$j(\rho) = \begin{cases} \rho & \text{if } \rho \in \mathcal{L} \\ -\alpha & \rho = \tau \\ 0 & \rho = \sigma \end{cases}$$

We then extend j by linearity. With this definition of j it is clear that

$$j \circ i_* = \text{id}: C_m(\mathcal{L}) \longrightarrow C_m(\mathcal{L}).$$

We have

$$P_{m-1}(\rho) = \begin{cases} -\sigma & \text{if } \rho = \tau \\ 0 & \text{otherwise.} \end{cases}$$

Now we check that

$$(\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m)(\rho) = (i_* \circ j)(\rho) - \rho.$$

If $\rho \in \mathcal{L}$ then both sides are zero, as expected. If $\rho = \sigma$ then

$$\begin{aligned}(\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m)(\sigma) &= P_{m-1}(\tau + \alpha) \\ &= -\sigma\end{aligned}$$

which is the same as the RHS, since $j(\sigma) = 0$. If $\rho = \tau$ then

$$\begin{aligned}(\partial_m \circ P_{m-1} + P_{m-2} \circ \partial_{m-1})(\tau) &= \partial_m P_{m-1}(\tau) \\ &= -\partial_m \sigma \\ &= -\tau - \alpha,\end{aligned}$$

which is the same as the RHS, since $j(\tau) = -\alpha$.

Thus $i_* \circ j$ is chain homotopic to the identity. As $j_* \circ i_*$ is the identity on homology, it is injective. Thus i_* is injective. As $i_* \circ j_*$ is the identity on homology, it is surjective. Thus i_* is surjective.

It follows that i_* is an isomorphism.

Challenge Problems: (Just for fun)

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