

MODEL ANSWERS TO THE FOURTH HOMEWORK

1. Suppose that X and Y are isomorphic. Then we may find morphisms

$$f: X \longrightarrow Y \quad \text{and} \quad g: Y \longrightarrow X$$

such that

$$g \circ f = \text{id}_X: X \longrightarrow X \quad \text{and} \quad f \circ g = \text{id}_Y: Y \longrightarrow Y.$$

If we apply F to f and g then we get morphisms

$$F(f): F(X) \longrightarrow F(Y) \quad \text{and} \quad F(g): F(Y) \longrightarrow F(X).$$

On the other hand if apply F to the equations above then we get

$$\begin{aligned} \text{id}_{F(X)} &= F(\text{id}_X) \\ &= F(g \circ f) \\ &= F(g) \circ F(f) \end{aligned}$$

and

$$\begin{aligned} \text{id}_{F(Y)} &= F(\text{id}_Y) \\ &= F(f \circ g) \\ &= F(f) \circ F(g). \end{aligned}$$

Thus $F(f)$ and $F(g)$ are isomorphisms and so X and Y are isomorphic.

2. (i) Suppose that X is an object of $\mathcal{C}$ and consider the identity morphism $\text{id}_X: X \longrightarrow X$. We want to show that

$$h_A(\text{id}_X) = \text{id}_{\text{Hom}(A, X)}.$$

Both sides are functions

$$\text{Hom}(A, X) \longrightarrow \text{Hom}(A, X),$$

and so it is enough to show they have the same effect on

$$\chi \in \text{Hom}(A, X).$$

If we are given a morphism $\chi: A \longrightarrow X$ then

$$\begin{aligned} h_A(\text{id}_X)(\chi) &= \text{id}_X \circ \chi \\ &= \chi \\ &= \text{id}_{\text{Hom}(A, X)}(\chi). \end{aligned}$$

Thus

$$h_A(\text{id}_X) = \text{id}_{\text{Hom}(A, X)}.$$

Now suppose that we are given three objects X , Y and Z , and two morphisms

$$f: X \longrightarrow Y \quad \text{and} \quad g: Y \longrightarrow Z.$$

We want to show that

$$h_A(g \circ f) = h_A(g) \circ h_A(f).$$

Both sides are functions

$$\text{Hom}(A, X) \longrightarrow \text{Hom}(A, Z),$$

and so it is enough to show they have the same effect on

$$\chi \in \text{Hom}(A, X).$$

If we are given a morphism $\chi: A \longrightarrow X$ then

$$\begin{aligned} h_A(g \circ f)(\chi) &= (g \circ f) \circ \chi \\ &= g \circ (f \circ \chi) \\ &= g \circ (h_A(f)(\chi)) \\ &= h_A(g)((h_A(f)(\chi))) \\ &= (h_A(g) \circ h_A(f))(\chi). \end{aligned}$$

Thus

$$h_A(g \circ f) = h_A(g) \circ h_A(f).$$

(ii) Suppose that X is an object of $\mathcal{C}$ and consider the identity morphism $\text{id}_X: X \longrightarrow X$. We want to show that

$$h^A(\text{id}_X) = \text{id}_{\text{Hom}(X, A)}.$$

Both sides are functions

$$\text{Hom}(X, A) \longrightarrow \text{Hom}(X, A),$$

and so it is enough to show they have the same effect on

$$\chi \in \text{Hom}(X, A).$$

If we are given a morphism $\chi: X \longrightarrow A$ then

$$\begin{aligned} h^A(\text{id}_X)(\chi) &= \chi \circ \text{id}_X \\ &= \chi \\ &= \text{id}_{\text{Hom}(X, A)}(\chi). \end{aligned}$$

Thus

$$h^A(\text{id}_X) = \text{id}_{\text{Hom}(X, A)}.$$

Now suppose that we are given three objects X , Y and Z , and two morphisms

$$f: X \longrightarrow Y \quad \text{and} \quad g: Y \longrightarrow Z.$$

We want to show that

$$h^A(g \circ f) = h^A(f) \circ h^A(g).$$

Both sides are functions

$$\text{Hom}(X, A) \longrightarrow \text{Hom}(Z, A),$$

and so it is enough to show they have the same effect on

$$\chi \in \text{Hom}(X, A).$$

If we are given a morphism $\chi: X \longrightarrow A$ then

$$\begin{aligned} h^A(g \circ f)(\chi) &= \chi \circ (g \circ f) \\ &= (\chi \circ g) \circ f \\ &= (h^A(g)(\chi)) \circ f \\ &= h^A(f)(h^A(g)(\chi)) \\ &= (h^A(f) \circ h^A(g))(\chi). \end{aligned}$$

Thus

$$h^A(g \circ f) = h^A(f) \circ h^A(g).$$

3. (i) Suppose that

$$\sigma = \langle v_0, v_1, \dots, v_n \rangle$$

Let

$$\tau = \langle v_1, v_2, \dots, v_n \rangle$$

Then

$$\sigma = \langle v_0, \tau \rangle.$$

Let $\mathcal{M}$ be the simplicial complex determined by τ (so that $\mathcal{M}$ is τ and all its faces).

Then $\mathcal{K} = \text{Cone}_{v_0}(\mathcal{M})$.

(ii) This follows from Theorem 9.3.

(iii) By Homework 3. Question 2, (iii), there are isomorphisms

$$H_m(\mathcal{L}) \simeq H_m(\mathcal{K}).$$

for $m \neq n$ or $n - 1$.

Otherwise Homework 3, Question 2, (v) implies there is an exact sequence

$$0 \longrightarrow H_n(\mathcal{L}) \xrightarrow{i_*} H_n(\mathcal{K}) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\pi} H_{n-1}(\mathcal{L}) \xrightarrow{i_*} H_{n-1}(\mathcal{K}) \longrightarrow 0.$$

Since

$$H_n(\mathcal{K}) = H_{n-1}(\mathcal{K}) = 0,$$

this implies that

$$H_n(\mathcal{L}) = 0 \quad \text{and} \quad H_{n-1}(\mathcal{L}) \simeq \mathbb{Z}.$$

Putting all of this together we have

$$H_m(\mathcal{L}) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \text{ or } m = n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

4. (i) Let $\sigma \in S(\mathcal{K})$ and let τ be a face of σ . Then either $\sigma \in \text{Cone}_{w_1}(\mathcal{K})$ in which case τ belongs to $\text{Cone}_{w_1}(\mathcal{K})$ as well or $\sigma \in \text{Cone}_{w_2}(\mathcal{K})$ in which case τ also belongs to $\text{Cone}_{w_2}(\mathcal{K})$. Either way τ belongs to $S(\mathcal{K})$.

Now suppose that σ and τ belong to $S(\mathcal{K})$ and let $\rho = \sigma \cap \tau$. Suppose that ρ is not empty. If σ and τ both belong to either $\text{Cone}_{w_1}(\mathcal{K})$ or $\text{Cone}_{w_2}(\mathcal{K})$ then ρ is a face of σ and τ . The only other possibility, possibly switching the roles of σ and τ , is that σ belongs to $\text{Cone}_{w_1}(\mathcal{K})$ and τ belongs to $\text{Cone}_{w_2}(\mathcal{K})$.

By assumption $\rho \subset \mathcal{K}$. Suppose that $\sigma = \langle w_1, \sigma_1 \rangle$ and $\tau = \langle w_2, \tau_2 \rangle$, where σ_1 and $\tau_2 \in \mathcal{K}$. Then

$$\rho = \sigma_1 \cap \tau_2.$$

It follows that ρ is a face of σ_1 and τ_2 . But then ρ is a face of σ and τ . Thus $S(\mathcal{K})$ is a simplicial complex.

(ii) We first show that we can embed $S^n(\mathcal{K})$ into $\mathbb{R}^{n+2}$. When $n = 0$ we have the 1-skeleton of a 2-simplex, which sits naturally in $\mathbb{R}^2$. Note that we can always place the centre of the 2-simplex at the origin.

If we know that

$$S^{n-1}(\mathcal{K}) \subset \mathbb{R}^{n+1}$$

we just place w_1 and w_2 at

$$w_1 = (0, 0, \dots, 0, 1) \quad \text{and} \quad w_2 = (0, 0, \dots, 0, -1) \in \mathbb{R}^{n+2}$$

and we embed $\mathbb{R}^{n+1}$ into $\mathbb{R}^{n+2}$ in the usual way.

Now we show that projection from the origin onto the sphere S^{n+1} defines a homeomorphism by induction on n .

This holds for $n = 0$, since projection from the origin to S^1 , of any triangle which contains the origin, is a homeomorphism.

It is easy then to see that any line through the origin of $\text{Cone}_{w_1}(\mathcal{K})$ (respectively $\text{Cone}_{w_2}(\mathcal{K})$) meets a unique point of the upper (respectively lower) hemisphere. It follows that projection from the origin defines a bijection from $S^n(\mathcal{K})$ to the sphere S^{n+1} . This is then a homeomorphism and this completes the induction.

(iii) Note that

$$\phi: C_m(\mathcal{K}) \longrightarrow C_{m+1}(S(\mathcal{K}))$$

is a group homomorphism. Suppose that

$$\alpha \in C_m(\mathcal{K})$$

is a boundary. Then we may find $\beta \in C_{m+1}(\mathcal{K})$ such that

$$\partial_{m+1}\beta = \alpha.$$

We have

$$\begin{aligned}\phi(\alpha) &= \langle w_1, \alpha \rangle - \langle w_2, \alpha \rangle \\ &= \langle w_1, \partial_{m+1}\beta \rangle - \langle w_2, \partial_{m+1}\beta \rangle \\ &= -\partial_{m+1}\langle w_1, \beta \rangle + \beta + \partial_{m+1}\langle w_2, \beta \rangle - \beta \\ &= \partial_{m+1}(-\langle w_1, \beta \rangle + \langle w_2, \beta \rangle).\end{aligned}$$

Thus $\phi(\alpha)$ is a boundary. If we compose ϕ with the natural map then

$$C_{m+1}(S(\mathcal{K})) \longrightarrow H_{m+1}(S(\mathcal{K}))$$

then we get a group homomorphism

$$C_m(\mathcal{K}) \longrightarrow H_{m+1}(S(\mathcal{K})).$$

By what we just showed the kernel of this map contains the boundaries $C_m(\mathcal{K})$. The universal property of the quotient induces a group homomorphism

$$\phi_*: H_m(\mathcal{K}) \longrightarrow H_{m+1}(S(\mathcal{K})).$$

(iv) We will show by induction on n that

$$H_m(S^n(\mathcal{K})) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \text{ or } m = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $n = 0$ then $S^0(\mathcal{K}) = \mathcal{K}$ is the 1-skeleton of a 2-simplex. We have already shown that

$$H_0(\mathcal{K}) = \mathbb{Z} \quad \text{and} \quad H_1(\mathcal{K}) = \mathbb{Z}$$

and that all other homology groups are zero. This is the case $n = 0$. Now suppose we know the homology groups of $S^{n-1}(\mathcal{K})$. Note that

$$H_0(S^n(\mathcal{K})) = \mathbb{Z}$$

as $S^n(\mathcal{K})$ is connected. Using ϕ_* it follows that the only other non-zero homology group is in dimension $n + 1$, when we get

$$H_{n+1}(S^n(\mathcal{K})) = \mathbb{Z}.$$

This gives us the homology groups of $S^n(\mathcal{K})$ and this completes the induction.