

MODEL ANSWERS TO THE THIRD HOMEWORK

1. (i) We start with the triangulation in class for the unit square I^2 which induces a triangulation of the annulus. Thus I^2 is divided into half by the line $y = 1/2$, into six by the lines $x = 1/3$ and $x = 2/3$ and then into twelve congruent triangles by adding six diagonals to each rectangle. Then we identify the two sides inside the lines $x = 0$ and $x = 1$ to get a triangulation $\mathcal{K}$ of the annulus.

On the other hand we divide the interval I into three equal pieces and identify 0 and 1 to get a triangulation of S^1 . In fact our triangulation is the 1-skeleton of a 2-simplex.

Consider the natural projection map

$$p: I^2 \longrightarrow I \quad \text{given by} \quad (x, y) \longrightarrow x.$$

p is continuous. If we compose this with the quotient map

$$I \longrightarrow |\mathcal{L}|$$

this gives a continuous map to $|\mathcal{L}|$

$$\pi: I^2 \longrightarrow |\mathcal{L}|.$$

Since the map

$$I^2 \longrightarrow |\mathcal{K}|$$

is a quotient map, the universal property of the quotient map gives a map

$$f: |\mathcal{K}| \longrightarrow |\mathcal{L}|,$$

which is easily seen to be simplicial.

Alternatively, there is a natural map of the abstractions of $\mathcal{K}$ and $\mathcal{L}$ and f is the associated simplicial map.

(ii) We compute the homology of the annulus. The annulus is connected and so

$$H_0(\mathcal{K}) \simeq \mathbb{Z}$$

with generator a vertex.

Let α be a 2-cycle. There are edges which belong to only one 2-simplex σ . The coefficient of α in σ must then be zero. This implies that the coefficient of six 2-simplices must be zero. This exposes six diagonals and by the same argument the coefficient of the six remaining simplices must be zero. Thus

$$H_2(\mathcal{K}) = 0.$$

We now turn to computing the first homology. As usual 1-cycles are the sum of 1-simplices whose vertices correspond to a circuit around the 1-skeleton. It is not hard to argue that any 1-cycle is a sum of three cycles. When we put back the 2-simplices the three cycles around the edges of a triangle are zero and all three cycles given by horizontal lines are homologous.

For the rest consider the pushforward map

$$f_*: H_1(\mathcal{K}) \longrightarrow H_1(\mathcal{L}).$$

If we label the vertices of the unit interval $w_0 = 0$, $w_1 = 1/3$ and $w_2 = 2/3$ then

$$H_1(\mathcal{L}) \simeq \mathbb{Z}$$

with generator the 1-cycle

$$\beta = \langle w_0, w_1 \rangle + \langle w_1, w_2 \rangle + \langle w_2, w_0 \rangle.$$

(Here we mix up the location of the vertices w_i). Note that β is not a boundary, since there are no 2-simplices. On the other hand, it is clear that every 1-cycle is a multiple of β .

$$H_1(\mathcal{L}) \simeq \mathbb{Z}$$

with generator β .

Suppose that we take the three vertices $v_0 = (0, 1)$, $v_1 = (1/3, 1)$ and $v_2 = (2/3, 1)$ in I^2 . Then the pushforward of the 1-cycle

$$\alpha = \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle$$

is β ,

$$f_*\alpha = \beta.$$

It follows that no multiple of α is zero, so that

$$H_1(\mathcal{K}) \simeq \mathbb{Z}$$

with generator α .

2. (i) Let τ be a simplex in $\mathcal{L}$ and let ρ be a face of τ . Then $\rho \in \mathcal{K}$. As σ is not a proper face of any simplex in $\mathcal{K}$ it follows that ρ is in $\mathcal{L}$. If τ_1 and τ_2 are two simplices in $\mathcal{L}$ then τ_1 and τ_2 are two simplices in $\mathcal{K}$. Thus $\tau_1 \cap \tau_2$ is either empty or a face of both τ_1 and τ_2 .

Thus $\mathcal{L}$ is a simplicial complex.

(ii) Let i be the natural inclusion. It is clear that i is a simplicial map.

(iii) It is clear that the natural map

$$i_*: C_j(\mathcal{L}) \longrightarrow C_j(\mathcal{K})$$

is an isomorphism, for $j \neq n$, since both $\mathcal{L}$ and $\mathcal{K}$ have the same j -simplices. Now consider the two relevant pieces of the chain complexes:

$$\begin{array}{ccccc} C_{j+1}(\mathcal{L}) & \xrightarrow{\partial_{j+1}} & C_j(\mathcal{L}) & \xrightarrow{\partial_j} & C_{j-1}(\mathcal{L}) \\ C_{j+1}(i) \downarrow & & C_j(i) \downarrow & & C_{j-1}(i) \downarrow \\ C_{j+1}(\mathcal{K}) & \xrightarrow{\partial_{j+1}} & C_j(\mathcal{K}) & \xrightarrow{\partial_j} & C_{j-1}(\mathcal{K}). \end{array}$$

If $j > n + 1$ or $j < n - 1$ then all vertical arrows are isomorphisms. In this case $C_j(i)$ induces a bijection between j -cycles and a bijection between j -boundaries. It is then clear we get the same homology. If $j = n + 1$ then the only vertical map which is not an isomorphism is the last $C_n(i)$. $C_n(i)$ is clearly injective.

We check that $C_{n+1}(i)$ induces a bijection between $(n + 1)$ -cycles. We know that there is a map

$$Z_{n+1}(i): Z_{n+1}(\mathcal{L}) \longrightarrow Z_{n+1}(\mathcal{K}).$$

It is injective as it is the restriction of $C_{n+1}(i)$. We just need to check it is surjective. Suppose that $\beta \in Z_{n+1}(\mathcal{K})$. Then we may find $\alpha \in C_{n+1}(\mathcal{L})$ such that

$$C_{n+1}(i)(\alpha) = \beta,$$

as $C_{n+1}(i)$ is surjective. We have

$$\begin{aligned} 0 &= \partial_{n+1}\beta \\ &= \partial_{n+1}(C_{n+1}(i)(\alpha)) \\ &= (\partial_{n+1} \circ C_{n+1}(i))(\alpha) \\ &= (C_n(i) \circ \partial_{n+1})(\alpha) \\ &= C_n(i)(\partial_{n+1}\alpha). \end{aligned}$$

As $C_n(i)$ is injective, it follows that

$$\partial_n\alpha = 0.$$

But then α is a cycle and $Z_{n+1}(i)$ is an isomorphism.

As the first two vertical maps are isomorphisms, it is clear that the boundaries are the same. Thus

$$i_*: H_{n+1}(\mathcal{L}) \longrightarrow H_{n+1}(\mathcal{K})$$

is also an isomorphism.

(iv) We just have to check that the image of

$$\mathbb{Z} \longrightarrow C_{n-1}(\mathcal{L})$$

lands in $Z_{n-1}(\mathcal{L})$. But this is the usual trick. Let

$$\beta = \partial_n\sigma.$$

If we compute

$$\partial_{n-1}i_*(\beta) = i_*(\partial_{n-1}\partial_n\sigma)$$

in $\mathcal{K}$ then we get zero. The computation of $\partial_{n-1}\beta$ does not change, even if we remove σ (equivalently i_* is injective). Thus

$$\partial_{n-1}\beta = 0,$$

in $\mathcal{L}$. Thus β is an $(n-1)$ -cycle in $\mathcal{L}$ and so we get a group homomorphism

$$\pi: \mathbb{Z} \longrightarrow H_{n-1}(\mathcal{L}).$$

(v) We just go left to right checking exactness. Exactness at $H_n(\mathcal{L})$ means that i_* is injective. We just have to check that

$$i_*: C_n(\mathcal{L}) \longrightarrow C_n(\mathcal{K})$$

is an isomorphism on boundaries. But this is clear since a boundary is the image of an $(n+1)$ -simplex and $\mathcal{K}$ and $\mathcal{L}$ have the same $(n+1)$ -simplices. Thus i_* is injective.

There is a group homomorphism

$$C_n(\mathcal{K}) \longrightarrow \mathbb{Z},$$

which just sends an n -cycle α to the coefficient of σ . This gives us a group homomorphism

$$Z_n(\mathcal{K}) \longrightarrow \mathbb{Z}.$$

Since σ is not a proper face of any simplex, the coefficient of σ in any boundary is zero. Thus the boundaries lie in the kernel of this map. The universal property of the quotient induces a group homomorphism

$$\phi: H_n(\mathcal{K}) \longrightarrow \mathbb{Z}.$$

Now we check exactness at $H_n(\mathcal{K})$. The map above induces an exact sequence

$$0 \longrightarrow C_n(\mathcal{L}) \longrightarrow C_n(\mathcal{K}) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Exactness just says that $\mathcal{K}$ has one more n -simplex σ than $\mathcal{L}$. If we restrict to cycles we lose exactness on the right

$$0 \longrightarrow Z_n(\mathcal{L}) \longrightarrow Z_n(\mathcal{K}) \longrightarrow \mathbb{Z}.$$

Quotienting out by boundaries we might lose exactness on the left (except that we already checked this does not happen) but we keep exactness at the middle, since we already saw that any boundary has coefficient zero

$$0 \longrightarrow H_n(\mathcal{L}) \longrightarrow H_n(\mathcal{K}) \longrightarrow \mathbb{Z}.$$

Thus we have exactness at $H_n(\mathcal{K})$.

Now we turn to exactness at $\mathbb{Z}$. Suppose we start with an n -chain $\alpha \in C_n(\mathcal{K})$. We can write this as

$$\alpha = a\sigma + i_*\beta,$$

where $\beta \in C_n(\mathcal{L})$. The image of this in $\mathbb{Z}$ is a and the image of this in $C_{n-1}(\mathcal{L})$ is $a\partial_n\sigma$.

If α is a cycle then

$$\begin{aligned} 0 &= \partial_n\alpha \\ &= a\partial_n\sigma + \partial_n(i_*\beta). \end{aligned}$$

But then

$$a\partial_n\sigma = \partial_n(-\beta)$$

is a boundary. It follows that $a\partial_n\sigma$ is homologous to zero in $C_{n-1}(\mathcal{L})$. Thus $\pi \circ \phi = 0$.

Now suppose that $\pi(a) = 0$. Then $a\partial_n\sigma$ is homologous to zero in $C_{n-1}(\mathcal{L})$. It follows that there is an n -cycle $\beta \in C_n(\mathcal{L})$ such that

$$\partial_n\beta = a\partial_n\sigma.$$

If we put

$$\alpha = a\sigma - i_*\beta$$

then

$$\begin{aligned} \partial_n\alpha &= \partial_n(a\sigma - i_*\beta) \\ &= a\partial_n\sigma - i_*\partial_n\beta \\ &= 0. \end{aligned}$$

Thus α is a cycle and $\phi(\alpha) = a$. Thus we have exactness at $\mathbb{Z}$.

Now we check exactness at $H_{n-1}(\mathcal{L})$. It is clear that the composition of π and i_* is zero, since 1 goes $\partial_n\sigma$ and

$$i_*(\partial_n\sigma) = \partial_n\sigma$$

which is a boundary in $C_{n-1}(\mathcal{K})$. Now suppose that $\beta \in Z_{n-1}(\mathcal{L})$ is an $(n-1)$ -cycle that is sent to zero in $H_{n-1}(\mathcal{K})$. Then $i_*\beta$ is homologous to zero in $C_{n-1}(\mathcal{K})$ and we may find $\gamma \in C_n(\mathcal{K})$ such that

$$\partial_n\gamma = i_*\beta.$$

We may write

$$\gamma = a\sigma + i_*\delta$$

for some $\delta \in C_n(\mathcal{L})$. Then

$$\phi(a) = a\sigma$$

is homologous to γ . Thus we have exactness at $H_{n-1}(\mathcal{L})$.

Finally we check that $H_{n-1}(i) = i_*$ is surjective. It is clear that

$$C_{n-1}(i): C_{n-1}(\mathcal{L}) \longrightarrow C_{n-1}(\mathcal{K})$$

is surjective, as they have the same $(n-1)$ -simplices. But if we start with an $(n-1)$ -cycle β in $Z_{n-1}(\mathcal{K})$ then we may find $\alpha \in C_{n-1}(\mathcal{K})$ such that

$$i_*\alpha = \beta.$$

As $\mathcal{K}$ and $\mathcal{L}$ have the same $(n-2)$ -simplices it follows that α is an cycle. Thus we have exactness at $H_{n-1}(\mathcal{K})$.

3 (i) This is just 2 (i), applied twice. The first time we apply 2 (i) we conclude that $\mathcal{M} = \mathcal{K} \setminus \{\sigma\}$ is a simplicial complex. But τ is a simplex of $\mathcal{M}$ which is not a face of any other simplex. Thus 2 (i) applied to $\mathcal{M}$ implies that

$$\begin{aligned} \mathcal{L} &= \mathcal{K} \setminus \{\sigma, \tau\} \\ &= \mathcal{M} \setminus \{\tau\} \end{aligned}$$

is a simplicial complex.

(ii) i is simply the composition of the two simplicial maps g from $\mathcal{L}$ to $\mathcal{M}$ and f from $\mathcal{M}$ to $\mathcal{K}$ given by 2 (ii). Thus i is a simplicial map as the composition of simplicial maps is simplicial.

(iii) By (ii)

$$\begin{aligned} i_* &= (f \circ g)_* \\ &= f_* \circ g_*. \end{aligned}$$

It suffices to check that f_* and g_* are isomorphisms.

For f note that

$$\pi: \mathbb{Z} \longrightarrow H_{n-1}(\mathcal{M})$$

is injective, as τ is not the face of any n -simplex in $\mathcal{M}$. Thus the exact sequence

$$0 \longrightarrow H_n(\mathcal{M}) \xrightarrow{f_*} H_n(\mathcal{K}) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\pi} H_{n-1}(\mathcal{M}) \xrightarrow{f_*} H_{n-1}(\mathcal{K}) \longrightarrow 0$$

breaks into two isomorphisms,

$$f_*: H_{n-1}(\mathcal{M}) \longrightarrow H_{n-1}(\mathcal{K}) \quad \text{and} \quad f_*: H_n(\mathcal{M}) \longrightarrow H_n(\mathcal{K}).$$

For g note that

$$\phi: H_{n-1}(\mathcal{M}) \longrightarrow \mathbb{Z}$$

is surjective as the coefficient of the the class of $\partial_n \sigma$ is ± 1 in τ . Once again this means that the exact sequence

$$0 \longrightarrow H_{n-1}(\mathcal{L}) \xrightarrow{g_*} H_{n-1}(\mathcal{M}) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\pi} H_{n-2}(\mathcal{L}) \xrightarrow{g_*} H_{n-2}(\mathcal{M}) \longrightarrow 0$$

breaks into two isomorphisms,

$$g_*: H_{n-2}(\mathcal{L}) \longrightarrow H_{n-2}(\mathcal{M}) \quad \text{and} \quad g_*: H_{n-1}(\mathcal{L}) \longrightarrow H_{n-1}(\mathcal{M}).$$

4. Take any of the usual triangulations $\mathcal{K}$ of the torus induced by a triangulation of I^2 with eighteen 2-simplices. Pick any 2-simplex σ and let

$$\mathcal{L} = \mathcal{K} \setminus \{\sigma\}.$$

We first compute the homomology of $\mathcal{L}$. If we remove a simplex σ_1 that shares an edge τ_1 with σ then we can delete both σ_1 and τ_1 to get a complex

$$\mathcal{M} = \mathcal{L} \setminus \{\sigma_1, \tau_1\}.$$

The homomology of $\mathcal{L}$ and $\mathcal{M}$ are the same by 3 (iii). Continuing in this way, removing a 2-simplex and then an exposed edge, we finally get down to a simplicial complex $\mathcal{M}$ that contains nothing but the sides of I^2 . This is a triangulation of the skeleton of two 2-simplices that share a common vertex.

It is easy to argue that the homomology of $\mathcal{M}$ is given by

$$H_0(\mathcal{M}) \simeq \mathbb{Z} \quad \text{and} \quad H_1(\mathcal{M}) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

A generator for the first group is a vertex and generators for the second group are the two cycles. All other homomology groups are zero.

As the homomology of $\mathcal{L}$ is the same as the homomology of $\mathcal{M}$, we have

$$H_0(\mathcal{L}) \simeq \mathbb{Z} \quad \text{and} \quad H_1(\mathcal{L}) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

A generator for the first group is a vertex and generators for the second group are a horizontal and a vertical cycle. All other homomology groups are zero. Now we have to track what happens when put back σ . The zeroth homomology group is the same and there is no non-zero homomology beyond the second homomology.

By 2 (v) there is an exact sequence

$$0 \longrightarrow H_2(\mathcal{K}) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} H_1(\mathcal{K}) \longrightarrow 0.$$

The first zero comes from the fact that the 2nd homomology of $\mathcal{L}$ is zero. Consider the map

$$\pi: \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}.$$

The boundary of σ is zero in $H_1(\mathcal{L})$. Thus π is the zero map and the exact sequence above breaks into two isomorphisms. Hence

$$H_0(\mathcal{K}) \simeq \mathbb{Z}, \quad H_1(\mathcal{K}) \simeq \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad H_2(\mathcal{K}) \simeq \mathbb{Z}.$$

Now we turn the real projective plane. We use the triangulation with six vertices. As before we delete a 2-simplex. Let's delete

$$\sigma = \langle v_4, v_5, v_6 \rangle.$$

Call the resulting simplicial complex $\mathcal{L}$.

From there on, we can play the same trick as before and delete pairs of edges and triangles without changing the homomology. This way we can erase all 2-simplices. For example, we can arrange to be left with the outer cycle, plus the three spokes that go radially into the centre of the circle. Then we can remove these three spokes and their isolated vertices, without changing the homomology. At the end we get the 1-skeleton $\mathcal{M}$ of a 2-simplex. Thus

$$H_0(\mathcal{M}) \simeq \mathbb{Z} \quad \text{and} \quad H_1(\mathcal{M}) \simeq \mathbb{Z}.$$

All other homomology groups are zero.

As the homomology of $\mathcal{L}$ is the same as the homomology of $\mathcal{M}$, we have

$$H_0(\mathcal{L}) \simeq \mathbb{Z} \quad \text{and} \quad H_1(\mathcal{L}) \simeq \mathbb{Z}.$$

All other homomology groups are zero.

Now we have to figure out what happens when we put back the 2-simplex σ . By 2 (v) there is an exact sequence

$$0 \longrightarrow H_2(\mathcal{K}) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z} \xrightarrow{i_*} H_1(\mathcal{K}) \longrightarrow 0.$$

Consider the map

$$\pi: \mathbb{Z} \longrightarrow \mathbb{Z}.$$

By direct computation the boundary of σ is equivalent in $\mathcal{L}$ to twice the generator of $\mathbb{Z}$. Thus π is multiplication by 2. Hence

$$H_0(\mathcal{K}) \simeq \mathbb{Z}, \quad H_1(\mathcal{K}) \simeq \mathbb{Z}_2 \quad \text{and} \quad H_2(\mathcal{K}) = 0.$$