

MODEL ANSWERS TO THE SECOND HOMEWORK

1. We compute each homomology group $H_m(\mathcal{K})$ in turn. Note first that these vanish as soon as $m > 2$.

We start with the second homomology group which is probably the most straightforward.

Let

$$\alpha = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$$

be a 2-cycle.

Let $\sigma = \sigma_i$ be one of the 2-simplices. Note that there is an edge e such that e is a side of σ and yet e is not a side of any other triangle.

It follows that the coefficient of e in $\partial_2\sigma$ is non-zero. As the coefficient of e in $\partial_2\alpha$ is zero, it follows that the coefficient of σ in α is zero.

As there was nothing special about σ , it follows that $\alpha = 0$. Therefore

$$H_2(\mathcal{K}) = 0.$$

Now let us turn to the zeroth homomology group. A general 0-cycle is of the form

$$\alpha = a_0v_0 + a_1v_1 + a_2v_3 + a_3v_4 + a_4v_4.$$

Any such has a degree

$$a_0 + a_1 + a_2 + a_3 + a_4.$$

The map

$$d: Z_0(\mathcal{K}) \longrightarrow \mathbb{Z}$$

which associates to a 0-cycle its degree is a group homomorphism. This map is clearly surjective.

Suppose that $\sigma = \langle v_i, v_j \rangle$ is a 1-simplex. Then $\partial_1\sigma = v_j - v_i$. This has degree zero. So the boundaries

$$B_1(\mathcal{K}) \subset \text{Ker } d.$$

In fact we have equality.

The easiest way to see this is to observe that two vertices connected by an edge are homologous (that is, they are equivalent modulo boundaries). It is easy to see that everything is equivalent to a multiple of one vertex in $H_0(\mathcal{K})$. But the class of a vertex is not zero, using the degree map. Thus

$$H_0(\mathcal{K}) = \mathbb{Z}.$$

It follows that

$$B_1(\mathcal{K}) = \text{Ker } d.$$

We finally turn to the first homomology. Let α be a 1-chain. The condition that α is a 1-cycle translates to the condition that one can rearrange the 1-simplices in the expansion of α so that α corresponds to a circuit in the 1-skeleton (which is a graph). Any circuit is a sum of cycles, a sequence of vertices and edges, where the only repeated vertices are the initial and last vertex. But any cycle is a sum of the 3-cycles around any triangle.

There are four such 3-cycles, but three of them bound a 2-simplex. If we take the boundary of the 2-simplex this kills that 3-cycle. There is one 3-cycle α left, the one that bounds part of the line $x = 1$.

I claim that α is a free generator of $H_1(\mathcal{K})$. Suppose that

$$\beta = a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2$$

and

$$m\alpha = \partial_2\beta.$$

Since the three simplices σ_0 , σ_1 and σ_2 bound each bound an edge bounded by no other 2-simplex and this edge has coefficient zero in α , it follows that $a_0 = a_1 = a_2 = 0$. Thus

$$H_1(\mathcal{K}) = \mathbb{Z}$$

generated by α .

2. Let's take the triangulation we used in class for the annulus. The Möbius band is connected and so

$$H_0(\mathcal{K}) = \mathbb{Z},$$

generated by the class of a vertex.

Now let us consider the 2-cycles

$$\alpha = \sum a_i\sigma_i.$$

Pick an edge e that belongs to only one 2-simplex σ . Then the coefficient of e in $\partial_2\sigma$ is non-zero. Thus the coefficient of σ in α is zero. This eliminates 6 of the 12 possible 2-simplices. But once we have eliminated any one of these 6, this immediately eliminates one more. Pick an edge f that belongs to two 2-simplices, σ and τ . If the coefficient of σ in α is zero then so is the coefficient of τ . Thus $\alpha = 0$ and so

$$H_2(\mathcal{K}) = 0.$$

Now let us turn to the 1-chains. Consider first 1-skeleton. As usual the 1-cycles are sums of cycles in the sense of graph theory. There are 12 of length 3 around the 12 triangles, one of length 3 (the horizontal line in the middle), call this α and one of length six (go along the top then go along the bottom).

When we put back the twelve 2-simplices we kill the first twelve cycles. It is not hard to argue that the six cycle is twice α , the horizontal cycle of length 3. We claim that α is a free generator of the first homomology. We have already seen that every 1-cycle is a multiple of α . Suppose that

$$m\alpha = \partial_2\beta.$$

As usual, the coefficients of the three 2-simplices at the top and the three 2-simplices at the bottom are zero, because they each have an edge shared by no other 2-simplex and these edges have coefficient zero in α . But then a similar argument using the six diagonal edges forces the other six 2-simplices to have coefficient zero.

Thus

$$H_1(\mathcal{K}) = \mathbb{Z}$$

generated by α .

The other homomology groups are zero.

3. This is a triangulation of the sphere, which is connected. In fact any two vertices are connected by a 1-simplex. Thus

$$H_0(\mathcal{K}) = \mathbb{Z}$$

with generator a vertex.

Now consider the 2nd homomology. Let

$$\alpha = a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$$

be a 2-cycle.

We first consider the possibility that one 2-simplex σ has coefficient zero in this sum. Pick an edge e belonging to σ . Let τ be the other 2-simplex that shares this edge e . Then the coefficient of e in $\delta_2\tau$ is non-zero. The only possibility is that the coefficient of τ in α is zero. Continuing in this way, it is not hard to show that the coefficient of every σ_i is zero, so that $\alpha = 0$.

Now suppose there are two 2-cycles, α and β . Suppose that the coefficient of σ in α is a and the coefficient of σ in β is b . Consider the 2-cycle

$$\gamma = b\alpha - a\beta.$$

By construction the coefficient of σ is zero in γ . By what we already showed $\gamma = 0$. Thus

$$b\alpha = a\beta.$$

All of this tells us what to look for. If there are any interesting 2-cycles then there must be one 2-cycle α with the property that every other 2-cycle is a multiple of α . No coefficients of α are zero and in fact it is reasonable to guess that the coefficients of α are ± 1 .

But the boundary α of the missing 3-simplex is the 2-cycle we are looking for. As expected its coefficients are ± 1 . Computing in the simplicial complex determined by the 3-simplex, we see that

$$\partial_2 \alpha = 0,$$

since every boundary is a cycle. But the computation does not change if we remove the 3-simplex and so

$$\partial_2 \alpha = 0,$$

in $\mathcal{K}$, so that α is a 2-chain. It is not the boundary of a 3-chain, since the group of all 3-chains is zero. Thus

$$H_2(\mathcal{K}) = \mathbb{Z}$$

with generator α .

Now consider the first homomology. The 1-skeleton is the complete graph on 4 vertices. Every cycle is a sum of three cycles. But when we put back the 2-simplices, these are all killed. Thus

$$H_1(\mathcal{K}) = 0.$$

The other homomology groups are zero.

4. We take the unit square I^2 and divide it into nine equal smaller size squares and then we put a diagonal into each of these smaller squares. We then identify the top and bottom sides and the two horizontal sides. As usual,

$$H_0(\mathcal{K}) \simeq \mathbb{Z},$$

generated by the class of a vertex.

Now let us consider the 2-cycles. Arguing as in (3), we look for a 2-cycle δ , with coefficients ± 1 . If this exists then every other 2-cycle is a multiple of δ . In fact if we flip the orientation of the 2-simplices with negative coefficient then we may assume all of the coefficients are 1. Let us orient every 2-simplex using the anticlockwise orientation in I^2 . Then every edge e gets an orientation from the two simplices it divides. It is an easy check to see that it gets the opposite orientation from each 2-simplex (including edges on the boundary of I^2).

Thus there is a 2-cycle δ , with coefficients 1. No multiple of δ is homologous to zero, as there are no 3-chains. It follows that

$$H_2(\mathcal{K}) \simeq \mathbb{Z}$$

with generator δ .

We now turn to calculating the first homomology. First we compute in the 1-skeleton. As usual, everything is a sum of three cycles. There are 18 from the triangles in I^2 and there are three from the horizontal lines and three from the vertical lines.

Putting back the 2-simplices kills the 18 three cycles coming from the triangles. It is not hard to check that all cycles coming from horizontal lines are homomologous. Similarly for the vertical lines. Let call the vertices along the top edge a_0, a_1, a_2 and the vertices $b_0 = a_0, b_1$ and b_2 along the left hand side. Put

$$\alpha = \langle a_0, a_1 \rangle + \langle a_1, a_2 \rangle + \langle a_2, a_0 \rangle \quad \text{and} \quad \beta = \langle b_0, b_1 \rangle + \langle b_1, b_2 \rangle + \langle b_2, b_0 \rangle$$

We have shown than ever 1-simplex is homomologous to a sum

$$a\alpha + b\beta.$$

It remains to show that none of these are a boundary. Suppose γ is a 2-simplex and

$$a\alpha + b\beta = \partial_2 \gamma.$$

Note we are free to adjust γ by a multiple of δ . Since the coefficients of δ are ± 1 we can arrange so that the coefficient of one of the 2-simplices σ_{18} at the bottom right corner is zero in $\gamma + d\delta$. On the other hand

$$\begin{aligned} \partial_2(\gamma + d\delta) &= \partial_2 \gamma + d\partial_2 \delta \\ &= \partial_2 \gamma \\ &= a\alpha + b\beta. \end{aligned}$$

Consider the diagonal edge e in the bottom, right hand square of the triangulation of I^2 . It belongs to two simplices. One of them is σ_{18} . Let's suppose that the other is σ_{17} . Since the coefficient of σ_{18} is zero in $\gamma + d\delta$ and the coefficient of e is zero in $a\alpha + b\beta$, it follows that the coefficient of σ_{17} is also zero.

Continuing in this way, eating away at the whole unit square, one can show that the coefficient of every 2-simplex in $\gamma + d\delta$ is zero (or what comes to the same thing, that γ is a multiple of δ). Thus

$$H_1(\mathcal{K}) \simeq \mathbb{Z} \oplus \mathbb{Z},$$

generated by α and β .

The other homomology groups are zero.

5. For some variation, let us compute in the triangulation given by the complete graph on 6 vertices. As usual,

$$H_0(\mathcal{K}) \simeq \mathbb{Z},$$

generated by the class of a vertex.

Now let us consider the 2-cycles. Arguing as in (3), we looking for a 2-cycle δ , with coefficients ± 1 . We are going to show that no such exists and it follows that every 2-cycle is zero.

Consider writing down a 2-cycle δ in I^2 . Then we are going to write down a sum of eighteen 2-simplices. The diagonal edges force the 18

triangles to have compatible orientations. This is in fact going to be the same choice as in question 4 (up to flipping all of the signs, or in other words, all of the orientations).

Since the edges at the sides are identified in opposite directions for the torus and the real projective plane, the very fact that this choice of δ in I^2 gives a 2-cycle on the torus is exactly the reason it does not give a 2-cycle in the real projective plane.

Thus

$$H_2(\mathcal{K}) = 0.$$

We now turn to calculating the first homomology. We first compute in the 1-skeleton. As before, everything is a sum of cycles. The ten 2-simplices kill all the cycles around the triangles. The 1-chain

$$\alpha = \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle$$

is obviously a cycle. It is not hard to see everything is a multiple of this cycle. Note that if we add α to itself this is the same as going around the outer circle.

With a little bit of work we can show that going around the outer circle is homologous to going around the inner circle. Indeed let β be

$$\langle v_1, v_5, v_4 \rangle + \langle v_1, v_4, v_2 \rangle + \langle v_2, v_4, v_3 \rangle + \langle v_3, v_6, v_1 \rangle + \langle v_1, v_6, v_2 \rangle + \langle v_2, v_5, v_3 \rangle + \langle v_3, v_5, v_1 \rangle.$$

If we compute

$$\partial_2 \beta$$

then we get the difference between the outer circle and the inner circle. Thus α is a generator of the first homomology and 2α is homologous to zero. But α is not a boundary, by the usual arguments. It follows that

$$H_1(\mathcal{K}) = \mathbb{Z}_2,$$

with generator α .

The other homomology groups are zero.

If we compute with a triangulation of I^2 something similar happens, although the computation is a little more involved.

Obviously the twist here is the fact that there is torsion in the first homomology.

One way to think about this is via the two to one cover of the real projective plane by the sphere S^2 . If we consider a geodesic that starts at the North pole and descends to the South pole then this defines a cycle in the real projective plane (for this we need to pick an appropriate triangulation). Twice this cycle comes from a cycle on the sphere, that goes to South pole and back. But the homomology of the sphere is zero, and so this cycle is zero, since it comes from (is the pushforward of) a cycle homologous to zero on the sphere.

Challenge Problems: (Just for fun)

6. The i th Betti number $b_i(\mathcal{K})$ of a simplicial complex $\mathcal{K}$ is the rank of the i th homology group $H_i(\mathcal{K})$ (recall that any finitely generated abelian group is isomorphic to $\mathbb{Z}^r \oplus T$, where T is finite. The rank r is simply the number of factors of $\mathbb{Z}$).

The **Euler characteristic** is then the alternating sum

$$b_0 - b_1 + b_2 - b_3 + \dots$$

(i) Let us suppose that the ranks of the chain complexes are $c_1, c_2, \dots$, so that

$$C_m(\mathcal{K}) \simeq \mathbb{Z}^{c_m}.$$

We want to write down a formula for the Betti numbers $b_1, b_2, \dots$.

The key result is that if we have a group homomorphism

$$\phi: \mathbb{Z}^a \longrightarrow \mathbb{Z}^b$$

then the rank n of the kernel of ϕ plus the rank r of the image of ϕ is equal to b ,

$$b = n + r.$$

There are two ways to see this. Recall that ϕ corresponds to an integer matrix $b \times a$. If we run elimination (with no division), then the resulting matrix is in Smith normal form. This means that the only non-zero entries are those on the main diagonal and each entry divides the next one down and across (note that everything divides zero).

The rank n of the kernel is the number of rows which are zero and the rank r of the image is the number of non-zero columns. But the i th column is non-zero if and only if the i th diagonal is non-zero if and only if the i th row is non-zero. The total number of rows is b , so that indeed

$$b = n + r.$$

Here is another, more sophisticated, way. We can tensor the groups $\mathbb{Z}^a$ and $\mathbb{Z}^b$ by $\mathbb{Q}$, to get a linear map

$$\pi: \mathbb{Q}^a \longrightarrow \mathbb{Q}^b.$$

Then the rank n of the kernel of ϕ and the rank r of the image of ϕ become the nullity and the rank of the linear map π . This uses the fact that the tensor product is in fact a functor (the group homomorphism ϕ becomes the linear map π), the fact that tensor product commutes over direct sum and the fact that

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q} \quad \text{and} \quad \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Q} \simeq 0.$$

In this case the result reduces to rank-nullity.

We now connect this to the Betti numbers. If we go from chains to cycles then we still get free groups, but now the rank is $k_1, k_2, \dots$ where k_m is the rank of the kernel of ∂_m . If we go from cycles to homology then

$$b_m = k_m - r_{m+1}$$

where r_{m+1} is the rank of ∂_{m+1} . Finally, we already observed that

$$c_m = r_m + k_m.$$

Putting all of this together we get

$$\begin{aligned} c_0 - c_1 + c_2 - c_3 + \dots &= (r_0 + k_0) - (r_1 + k_1) + (r_2 + k_2) - (r_3 + k_3) + \dots \\ &= (k_0 - r_1) + (r_2 - k_1) + (k_2 - r_3) + \dots \\ &= (k_0 - r_1) + (k_1 - r_2) + (k_2 - r_3) + \dots \\ &= b_0 - b_1 + b_2 + \dots \\ &= \chi(\mathcal{K}). \end{aligned}$$

(ii)

We have

$$\begin{array}{ll} \chi(\mathcal{K}_1) = 5 - 8 + 3 = 0 & \text{and} \quad \chi(\mathcal{K}_1) = 1 - 1 + 0 - 0 \\ \chi(\mathcal{K}_2) = 9 - 14 + 6 = 1 & \text{and} \quad \chi(\mathcal{K}_2) = 1 - 1 + 1 = 1 \\ \chi(\mathcal{K}_3) = 4 - 6 + 4 = 2 & \text{and} \quad \chi(\mathcal{K}_3) = 1 - 0 + 1 = 2 \\ \chi(\mathcal{K}_4) = 9 - 27 + 18 = 0 & \text{and} \quad \chi(\mathcal{K}_4) = 1 - 2 + 1 = 0 \\ \chi(\mathcal{K}_5) = 6 - 15 + 10 = 1 & \text{and} \quad \chi(\mathcal{K}_5) = 0 - 0 + 1 = 1. \end{array}$$

Note that for the first homework there was a mistake. One does not get a triangulation of the real projective plane using the same triangulation for the torus or Klein bottle, unless one chooses the triangulation very carefully. The diagonals in the squares have to go through the corners and we identify opposite corners.

The model answers have been updated accordingly.