

MODEL ANSWERS TO THE FIRST HOMEWORK

1. Let us prove a third hybrid statement and reduce (a) and (b) to this statement.

We suppose that $m \leq n$.

The following are equivalent:

(I) The points $q_0, q_1, \dots, q_m$ span an affine space Λ of dimension m .

(II) The vectors $w_1, w_2, \dots, w_m$ are independent, where

$$w_i = q_i - q_0.$$

Suppose that (I) holds. Let $V = \Lambda - q_0$. Then V is a linear subspace of dimension m . As the vectors $w_1, w_2, \dots, w_m$ span V it follows that they are independent. Thus (II) holds.

Now suppose that (II) holds. Let V be the span of the vectors $v_1, v_2, \dots, v_m$.

Then $v_1, v_2, \dots, v_m$ is a basis for V and so V has dimension m . But then $\Lambda = V + q$ has dimension m . Thus (I) holds.

We now show that (b) holds.

If (i) holds then the $(m+1)$ points must span an affine linear space of dimension m . Thus (ii) holds.

If (ii) holds then we already saw that (iii) holds.

Now suppose that (iii) holds. Let $q_0, q_1, \dots, q_k$ be any subset of the points $p_0, p_1, \dots, p_m$. Note that if we put

$$u_i = v_i + (p_0 - q_0)$$

then the vectors $u_1, u_2, \dots, u_m$ are independent, by the equivalence of (I) and (II). Thus we may assume that $p_0 = q_0$. Let $w_i = q_i - q_0$. As the vectors $v_1, v_2, \dots, v_m$ are independent it follows that the vectors $w_1, w_2, \dots, w_k$ are independent. But then the points $q_0, q_1, \dots, q_k$ span a k -plane. Thus (i) holds.

Thus (b) holds.

Now we turn to (a). Let $q_0, q_1, \dots, q_n$ be any subset of $n+1$ points. Then (i) holds if and only if these points are always in linear general position. But then the results holds by (b) applied to every choice of points $q_0, q_1, \dots, q_n$.

2. Let

$$\{C_i \mid i \in J\}$$

be a collection of convex subsets of $\mathbb{R}^n$. Let C be their intersection,

$$C = \bigcap_{i \in J} C_i.$$

Suppose that p and $q \in C$ and $t \in [0, 1]$. As C_i is convex, it follows that

$$tp + (1 - t)q \in C_i.$$

But then

$$tp + (1 - t)q \in \bigcap_{i \in J} C_i = C.$$

Thus C is convex.

Now consider the set of all convex sets that contain X ,

$$\{X \subset D \mid D \text{ is convex}\}.$$

Then the intersection C is convex and so it is the smallest convex set containing X .

3. $2^n - 1$. Indeed, giving a face is the same as giving a subset of the vertices, apart from the empty set. But the powerset of a set with n elements has 2^n elements. Excluding the empty set gives us $2^n - 1$ elements.

4. We have to check two things. Suppose that $\sigma \in \mathcal{L}$ and that τ is a face of σ . Then $\sigma \in \mathcal{K}$ so that $\tau \in \mathcal{K}$. As $\sigma \in \mathcal{L}$, it follows that σ has dimension at most k . As τ is a face of σ , it follows that τ has dimension at most k . Thus $\tau \in \mathcal{K}$.

Now suppose that σ and $\tau \in \mathcal{L}$ and that $\rho = \sigma \cap \tau$ is not empty. Then σ and $\tau \in \mathcal{K}$ and so ρ is a face of both σ and τ .

Thus $\mathcal{L}$ is a simplicial complex.

5. (a) Perhaps the easiest triangulation is given by taking the 2-skeleton $\mathcal{L}$ of the simplicial complex given by a tetrahedron and all its faces. It is not hard to see that $|\mathcal{L}|$ is homeomorphic to S^2 , for example by projection from the centre of the tetrahedron onto a sphere of sufficiently large radius.

For the torus, take the unit square and divide it into nine squares, by inserting two vertical line segments (of the lines $x = 1/3$ and $x = 2/3$ and two horizontal line segments (of the lines $y = 1/3$ and $y = 2/3$). Now insert nine diagonals, one for each of the nine squares. This gives a triangulation of I^2 .

To see that we get a triangulation of the torus, we can proceed in three ways. We can take three triangles and cross them with I to get three surfaces, whose sides look like the sides of a bar of toblerone. Then we can arrange these in a triangle in space and identify open sides (this will involve a little bit of woodworking to get the triangles to match up). Adding a diagonal to all nine sides gives a triangulation.

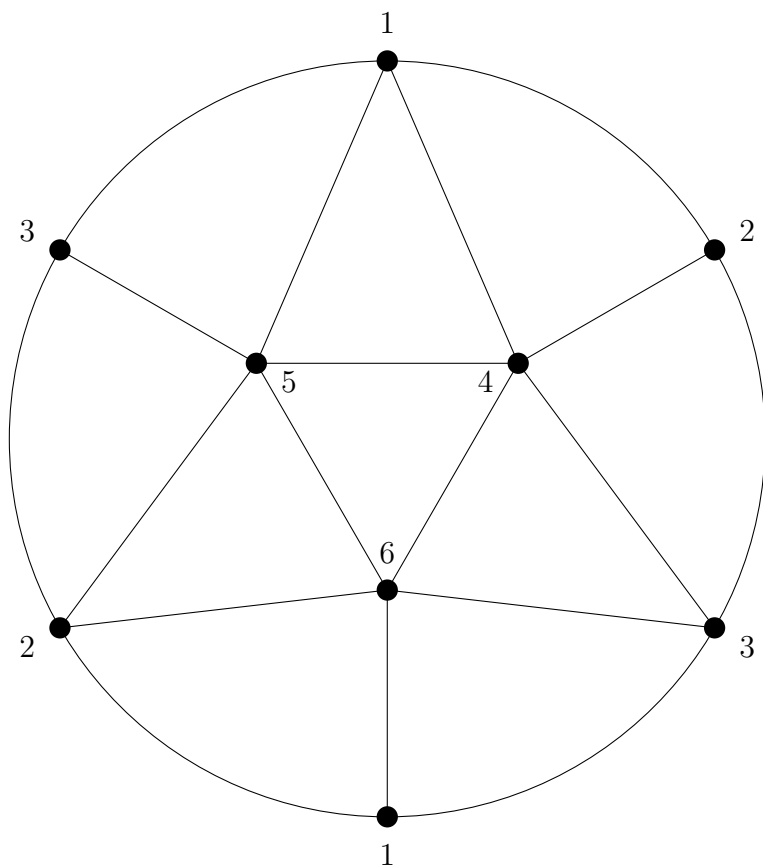
Or we can start with the standard torus and reverse everything. The vertices in I^2 get mapped to points of the torus. The vertical lines are mapped to three of the small circles and the image of the horizontal

lines are mapped to three of the long circles. Each square gets mapped to a rectangular region on the torus. We can just individually flatten each such region so it is a rectangle in a plane. Adding a diagonal to each of the nine rectangles gives a triangulation.

Perhaps the easiest is to just to use the fact that the quotient map describe an abstract simplicial complex. Then just use the fact that any abstract simplicial complex can be realised as a simplicial complex. The same triangulation will work for the Klein bottle.

20/04/25: The following corrects some mistakes from last week. The same triangulation does **not** always work for the real projective plane. The problem is at the corners. The antipodal map identifies opposite corners. Two sides at one corner are matched up with two sides at the antipodal corner. But if two triangles share two sides then they are the same triangle (if they share two sides, they share three vertices and a triangle is determined by its vertices).

To get around this, one should use a triangulation with the property that at each corner the diagonal goes through the corner. In this case one does indeed get a triangulation. Note that the corners map to two distinct points and not one point (which is what happens for the torus). However, probably, the easiest fix is just to use a more efficient triangulation. Triangulations of the real projective plane are interesting as there is a triangulation which corresponds to the complete graph on six vertices. The notion of linear general position works in projective space. If one starts with six points in linear general position (meaning no three points are collinear, that is no three points lie in a projective line) and take the resulting resulting way to join any two points and a triangle with any three points then one gets a triangulation.



(b) For $\mathcal{L}$ we have $v = 4$, $e = 6$ and $f = 4$. Thus the Euler characteristic is

$$v - e + f = 4 - 6 + 4 = 2.$$

Consider the triangulation of I^2 . There are 16 vertices, 33 vertices and 18 faces.

When we take the quotient down to $S^1 \times S^1$ we identify four pairs of vertices and one quadruple of vertices to a single vertex. Thus $v = 16 - 4 - 3 = 9$. We also identify 6 pairs of edges. Thus $e = 33 - 6 = 27$. Finally we don't identify any faces. Thus $f = 18$. Thus the Euler characteristic is

$$v - e + f = 9 - 27 + 18 = 0.$$

The same computation works for the Klein bottle.

For the real projective plane for the triangulation in the picture we have $v = 6$, $e = 15$ and $f = 10$. The Euler characteristic is

$$v - e + f = 6 - 15 + 10 = 1.$$

Or if we use a triangulation of I^2 note that the other only thing that changed is that there is one less identification of vertices. We get

$v = 10$. The other numbers are the same and we get

$$v - e + f = 10 - 27 + 18 = 1.$$

6. (i) We have to check that the composition of two morphisms is a morphism and that there is an identity morphism.

Given two morphisms

$$f: (V_X, X) \longrightarrow (V_Y, Y) \quad \text{and} \quad g: (V_Y, Y) \longrightarrow (V_Z, Z)$$

between three abstract simplicial complexes, define the composition

$$h = g \circ f: (V_X, X) \longrightarrow (V_Z, Z)$$

by simply composing the two maps on vertices

$$h = g \circ f: V_X \longrightarrow V_Z$$

We check that this is a morphism. Suppose that $\sigma \in X$. Then $f(\sigma) \in Y$, as f is a morphism. It follows that

$$h(\sigma) = g(f(\sigma)) \in Z,$$

as g is a morphism. Thus we can compose morphisms. Composition is associative as composition of functions is associative.

Finally consider

$$i_{(V_X, X)}: (V_X, X) \longrightarrow (V_X, X)$$

given by the identity function $i_{V_X}: V_X \longrightarrow V_X$. This is clearly a morphism and it clearly plays the role of the identity morphism. Thus we have a category.

(ii) Suppose that $T \subset \Sigma$, where $\Sigma \in X$. Thus there is a simplex σ such that the elements of Σ are the vertices of σ . As the elements of T are a subset of the vertices of σ it follows that there is a face τ of σ whose vertices are the elements of T . As $\mathcal{K}$ is a simplicial complex, it follows that $\tau \in \mathcal{K}$. But then $T \in X$.

Thus (V_X, X) is an abstract simplicial complex.

(iii) Suppose that

$$F: (V_X, X) \longrightarrow (V_Y, Y)$$

is an isomorphism. Then $\Sigma \in X$ if and only if $F(\Sigma) \in Y$.

Note that if we order the vertices V_X then F automatically induces an ordering of the vertices V_Y .

Suppose that $\sigma \in \mathcal{K}$. Let $\Sigma \in X$ be the set of vertices of σ and let $T = F(\Sigma) \in Y$. The $\tau \in \mathcal{L}$ be the simplex corresponding to T .

Define a homeomorphism

$$f_\sigma: |\mathcal{K}| \longrightarrow |\mathcal{L}|$$

using barycentric coordinates, with the given orderings of the vertices of σ and τ .

Suppose that $\rho = \sigma \cap \tau$ is not empty. Then ρ is a face of both. In this case

$$f_\sigma|_\rho = f_\tau|_\rho$$

simply because these two functions use the same barycentric coordinates on ρ and its image.

As the simplices σ are closed subsets of $|\mathcal{K}|$ it follows that we can patch together these functions to a function

$$f: |\mathcal{K}| \longrightarrow |\mathcal{L}|.$$

This function is continuous and a bijection. It follows easily that it is a homeomorphism.

Challenge Problems: (Just for fun)

(iv) There are two completely different ways to proceed.

One way is to put a probability measure on $\mathbb{R}^n$, with property that any proper linear subspace has measure zero (there are lots of these). We construct a sequence of points

$$p_1, p_2, \dots$$

in linear general position, as follows. Suppose that we have constructed $p_1, p_2, \dots, p_m$. Take the union Z of the span of all subsets with at most n elements. Then Z is a finite union of proper linear subspaces. Therefore it has measure zero. So there is a point p_{m+1} not belonging to Z . But then $p_1, p_2, \dots, p_{m+1}$ is in linear general position.

Here is a completely different way. Consider the map

$$\mathbb{R} \longrightarrow \mathbb{R}^n \quad \text{given by} \quad t \longrightarrow (t, t^2, t^3, \dots, t^n).$$

The image is a curve C . Pick infinitely many points on C by picking infinitely many points

$$\lambda_1, \lambda_2, \dots \in \mathbb{R}.$$

For example we could take the natural numbers $\mathbb{N} \subset \mathbb{R}$. I claim that the corresponding points

$$\{(\lambda_i, \lambda_i^2, \lambda_i^3, \dots, \lambda_i^n) \mid i \in \mathbb{N}\}$$

are in linear general position.

For this consider the Vandermonde determinant,

$$\det A \quad \text{where} \quad a_{ij} = \lambda_i^j.$$

Here the indices i and j both run from 0 to n . It is known that this determinant is

$$\prod_{i < j} (\lambda_i - \lambda_j).$$

In particular it is non-zero if the $\lambda_0, \lambda_1, \dots, \lambda_n$ are distinct. If we subtract the first column from the other n columns this tells us the n vectors

$$(\lambda_i - \lambda_0, \lambda_i^2 - \lambda_0^2, \lambda_i^3 - \lambda_0^3, \dots, \lambda_i^n - \lambda_0^n)$$

are independent. But then the points we wrote down are in linear general position.

(v) Simply take $m = |V_X|$ points in $\mathbb{R}^{2n+1}$ which are in linear general position. The existence of these points is guaranteed by (iv). If $\Sigma \in X$ then let σ be the simplex generated by the vertices in Σ .

We check that this gives us a simplicial complex $\mathcal{K}$. Suppose that τ is a face of $\sigma \in \mathcal{K}$. Let T be the set of vertices of τ . Then $T \subset \Sigma$ and so $T \in X$. Thus $\tau \in \mathcal{K}$, by definition of $\mathcal{K}$.

Now suppose that σ and $\tau \in \mathcal{K}$. Let Σ and T be the set of vertices of σ and τ . Let s be the cardinality of Σ , let t be the cardinality of T and let r be the cardinality of the intersection P . Let Λ , Λ_Σ , Λ_T , and Λ_R be the span of the sets $\Sigma \cup T$, Σ , T and P . Then Λ_Σ , Λ_T , and Λ_R have dimension $s - 1$, $t - 1$ and $r - 1$. Λ contains $s + t - r$ points of V_X and so Λ has dimension at most $s + t - r - 1$.

On the the other hand

$$\begin{aligned} \dim \Lambda &= \dim \Lambda_\Sigma + \dim \Lambda_T - \dim \Lambda_R \\ &= (s - 1) + (t - 1) - (r - 1) \\ &= s + t - r - 1. \end{aligned}$$

Thus Λ must contain exactly $s + t$ points of V_X and Λ_R must contain exactly r points of V_X . It follows that

$$\sigma \cap \tau = \rho$$

is the span of the points R . Thus ρ is a face of both σ and τ .

Thus $\mathcal{K}$ is a simplicial complex. It is clear that its abstraction is (V_X, X) .