

SECOND MIDTERM
MATH 190B, UCSD, SPRING 25

You have 50 minutes.

There are 4 problems, and the total number of points is 55. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:_____

Signature:_____

Student ID #:_____

Problem	Points	Score
1	15	
2	10	
3	10	
4	20	
5	10	
6	10	
Total	55	

1. (15pts) *Give the definition of a chain homotopy.*

A chain homotopy between two simplicial maps f and g of two simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ is a sequence of group homomorphisms

$$P_m: C_m(\mathcal{K}) \longrightarrow C_{m+1}(\mathcal{L})$$

such that

$$\delta_{m+1} \circ P_m + P_{m-1} \circ \delta_m = f_* - g_*.$$

(ii) *Give the definition of a simplicial approximation.*

If

$$h: |\mathcal{K}| \longrightarrow |\mathcal{L}|$$

is a continuous map of simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ then a simplicial map f from $\mathcal{K}$ to $\mathcal{L}$ is an approximation of h if for every vertex v of $\mathcal{K}$

$$h(\text{St}(v)) \subset \text{St}(f(v)).$$

(iii) *Give the definition of the Barycentre.*

If

$$\sigma = \langle v_0, v_1, \dots, v_m \rangle$$

is an m -simplex then the Barycentre of σ is the point

$$\hat{\sigma} = \frac{v_0 + v_1 + \dots + v_m}{m+1}.$$

2. (10pts) *Show that if F is a (covariant) functor from the category $\mathcal{C}$ to the category $\mathcal{D}$ and X and Y are isomorphic objects of $\mathcal{C}$ then $F(X)$ and $F(Y)$ are isomorphic objects of $\mathcal{D}$.*

By assumption we may find morphisms

$$f: X \longrightarrow Y \quad \text{and} \quad g: Y \longrightarrow X$$

such that

$$g \circ f = \text{id}_X: X \longrightarrow X \quad \text{and} \quad f \circ g = \text{id}_Y: Y \longrightarrow Y.$$

If we apply F to f and g then we get morphisms

$$F(f): F(X) \longrightarrow F(Y) \quad \text{and} \quad F(g): F(Y) \longrightarrow F(X).$$

If we apply F to the first equation then we get

$$\begin{aligned} \text{id}_{F(X)} &= F(\text{id}_X) \\ &= F(g \circ f) \\ &= F(g) \circ F(f). \end{aligned}$$

Similarly if we apply F to the second equation then we get

$$\begin{aligned} \text{id}_{F(Y)} &= F(\text{id}_Y) \\ &= F(f \circ g) \\ &= F(f) \circ F(g). \end{aligned}$$

Thus $F(f)$ and $F(g)$ are inverses, so that $F(f)$ and $F(g)$ are isomorphisms and $F(X)$ and $F(Y)$ are isomorphic.

3. (10pts) Suppose that f and g are two simplicial maps from the simplicial complex $\mathcal{K}$ to the simplicial complex $\mathcal{L}$. If f and g are chain homotopic then show that they induce the same map on homology.

By assumption there is a chain homotopy from f to g , a sequence of group homomorphisms

$$P_m: C_m(\mathcal{K}) \longrightarrow C_{m+1}(\mathcal{L})$$

such that

$$\delta_{m+1} \circ P_m + P_{m-1} \circ \delta_m = f_* - g_*.$$

Suppose that $\alpha \in Z_m(\mathcal{K})$ is an m -cycle. We have

$$\begin{aligned} f_*\alpha - g_*\alpha &= (f_* - g_*)\alpha \\ &= (\delta_{m+1} \circ P_m + P_{m-1} \circ \delta_m)(\alpha) \\ &= \delta_{m+1}(P_m\alpha) + P_{m-1}(\delta_m\alpha) \\ &= \delta_{m+1}\beta, \end{aligned}$$

where

$$\beta = P_m(\alpha) \in C_{m+1}(\mathcal{L}).$$

But then $f_*\alpha$ and $g_*\alpha$ are homologous, so that their image in $H_m(\mathcal{L})$ is equal. It follows that

$$f_* = g_*: H_m(\mathcal{K}) \longrightarrow H_m(\mathcal{L}).$$

4. (20pts) (i) Show that if $\mathcal{K} \subset \mathbb{R}^N$ is a simplicial complex and w is a point of $\mathbb{R}^N$ such that every line through w meets $\mathcal{K}$ in at most one point then $\mathcal{L} = \text{Cone}_w(\mathcal{K})$ is a simplicial complex.

Suppose that

$$\sigma = \langle v_0, v_1, \dots, v_m \rangle$$

is a simplex in $\mathcal{K}$. Then w does not lie in the affine space spanned by σ and so

$$w, v_0, v_1, \dots, v_m$$

span an affine space of dimension $m + 1$. Thus $\langle w, \sigma \rangle$ is an $(m + 1)$ -simplex.

The simplices μ of $\mathcal{L}$ are of three forms

- (1) $\sigma \in \mathcal{K}$,
- (2) $\langle w, \sigma \rangle$, where $\sigma \in \mathcal{K}$.
- (3) w .

In particular it is clear that if $\mu \in \text{Cone}_w(\mathcal{K})$ and ν is a face of μ then $\nu \in \text{Cone}_w(\mathcal{K})$.

Now suppose that σ and τ are two simplices in $\text{Cone}_w(\mathcal{K})$. Suppose $\rho = \sigma \cap \tau$ is non-empty. If σ and τ are simplices in $\mathcal{K}$ then ρ is certainly a face of both σ and τ .

If $\sigma = \langle w, \sigma_0 \rangle$, where σ_0 and τ are simplices in $\mathcal{K}$ then

$$\rho = \sigma_0 \cap \tau.$$

Then ρ is a face of both σ_0 and τ , so that ρ is also a face of σ .

Similarly if

$$\sigma = \langle w, \sigma_0 \rangle \quad \text{and} \quad \tau = \langle w, \tau_0 \rangle$$

where σ_0 and τ_0 are simplices in $\mathcal{K}$ then

$$\rho_0 = \sigma_0 \cap \tau_0,$$

is a face of both σ_0 and τ_0 , so that

$$\rho = \langle w, \rho_0 \rangle$$

is a face of σ and τ . It follows that $\text{Cone}_w(\mathcal{K})$ is a simplicial complex.

(ii) Show that the reduced homology of $\mathcal{L}$ is zero,

$$\tilde{H}_m(\mathcal{L}) = 0.$$

$\mathcal{L}$ is connected and so $\tilde{H}_0(\mathcal{L}) = 0$.

The identity map i from the simplicial complex $\mathcal{L}$ to $\mathcal{L}$ induces the identity on the support and the identity on chains

$$i_*: C_m(\mathcal{L}) \longrightarrow C_m(\mathcal{L}).$$

We show that i_* is chain homotopic to the zero map. We define a group homomorphism

$$P_m: C_m(\mathcal{L}) \longrightarrow C_{m+1}(\mathcal{L}).$$

It suffices to define P_m on simplices. If σ is an m -simplex then we put

$$P_m(\sigma) = \begin{cases} \langle w, \sigma \rangle & w \text{ is not a vertex of } \sigma \\ 0 & \text{otherwise.} \end{cases}$$

We then extend P_m by linearity. It is convenient to use the following notation:

$$\langle w, \alpha \rangle = P_m(\alpha).$$

We check that

$$\partial_{m+1} \langle w, \alpha \rangle = \alpha - \langle w, \partial_m \alpha \rangle.$$

It is enough to check this on m -simplices σ . If σ does not contain w , it is an easy consequence of the definition of ∂_{m+1} . If σ does contain w then the LHS is zero and we need to check the RHS is zero. We may assume that the first entry of σ is w . In this case

$$\langle w, \partial_m \sigma \rangle = \sigma,$$

since the only term that survives when we compute $\partial_m \sigma$ is when we drop w . Thus the RHS is also zero.

We now compute

$$\begin{aligned} (\delta_{m+1} \circ P_m + P_{m-1} \circ \delta_m)(\alpha) &= \partial_{m+1} \langle w, \alpha \rangle + \langle w, \partial_m \alpha \rangle \\ &= \alpha \\ &= i_* \alpha - 0. \end{aligned}$$

Thus the identity and the zero map are the same on the reduced homology. This is only possible if the reduced homology is zero.

Bonus Challenge Problems

5. (10pts) *Compute the homology of the sphere S^n (you may use results from the homework that relate the homology of a simplicial complex and the homology you get if you delete a simplex, if you state these results carefully).*

We start with the complex $\mathcal{K}$ determined by an n -simplex σ . It is easy to see that $\mathcal{K}$ is a cone over any vertex of σ . It follows by 4 (ii) that the reduced homology of $\mathcal{K}$ is zero.

Now let $\mathcal{L}$ be the simplicial complex you get by deleting σ . Then $|\mathcal{L}|$ is homeomorphic to S^{n-1} . Indeed place σ into $\mathbb{R}^n$ by putting the origin in the interior and project from the origin. This implies that the boundary $|\mathcal{L}|$ is homeomorphic to S^{n-1} .

Let i be the simplicial map from $\mathcal{L}$ to $\mathcal{K}$ given by the natural inclusion. This induces a map on homology

$$i_*: H_m(\mathcal{L}) \longrightarrow H_m(\mathcal{K}).$$

Recall that i_* is an isomorphism outside of the range $i = n$ and $i = n-1$ and that there is an exact sequence

$$0 \longrightarrow H_n(\mathcal{L}) \xrightarrow{i_*} H_n(\mathcal{K}) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\pi} H_{n-1}(\mathcal{L}) \xrightarrow{i_*} H_{n-1}(\mathcal{K}) \longrightarrow 0.$$

Since the homology groups of $\mathcal{K}$ are all zero, for $m > 0$, it follows that

$$H_n(\mathcal{L}) = 0,$$

and the rest of the exact sequence above reduces to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\pi} H_{n-1}(\mathcal{L}) \longrightarrow 0.$$

Thus

$$H_{n-1}(\mathcal{L}) \simeq \mathbb{Z}.$$

Putting all of this together we have

$$H_m(S^n) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \text{ or } m = n \\ 0 & \text{otherwise.} \end{cases}$$

6. (10pts) Let $\mathcal{K}$ be a simplicial complex and let σ and $\tau \in \mathcal{K}$ be two simplices, such that σ is not a proper face of a simplex in $\mathcal{K}$ and τ is a face of σ that is not a proper face of any other simplex in $\mathcal{K}$. Let $\mathcal{L} = \mathcal{K} \setminus \{\sigma, \tau\}$.

Show that

$$i_*: C_m(\mathcal{L}) \longrightarrow C_m(\mathcal{K})$$

is a chain equivalence.

We want to define a group homomorphism j and a chain homotopy P ,

$$j: C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{L}) \quad \text{and} \quad P_m: C_m(\mathcal{K}) \longrightarrow C_{m+1}(\mathcal{K}).$$

We define j be the identity on $C_m(\mathcal{L}) \subset C_m(\mathcal{K})$ and P_m to be zero. Since τ is a face of σ it follows that

$$\partial_m \sigma = a\tau + \alpha,$$

where $a = \pm 1$ and α is not supported on τ . Possibly flipping the orientation of τ we may assume that $a = 1$. We put $P_m(\sigma) = 0$. Similarly we put $j(\sigma) = 0$. After some experimentation we put $P_{m-1}(\tau) = \sigma$ and $j(\tau) = -\alpha$.

Putting all of this together we have

$$j(\rho) = \begin{cases} \rho & \text{if } \rho \in \mathcal{L} \\ -\alpha & \rho = \tau \\ 0 & \rho = \sigma. \end{cases}$$

We then extend j by linearity. With this definition of j it is clear that

$$j \circ i_* = \text{id}: C_m(\mathcal{L}) \longrightarrow C_m(\mathcal{L}).$$

We have

$$P_{m-1}(\rho) = \begin{cases} -\sigma & \text{if } \rho = \tau \\ 0 & \text{otherwise.} \end{cases}$$

Now we check that

$$(\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m)(\rho) = (i_* \circ j)(\rho) - \rho.$$

If $\rho \in \mathcal{L}$ then both sides are zero, as expected. If $\rho = \sigma$ then

$$\begin{aligned} (\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m)(\sigma) &= P_{m-1}(\tau + \alpha) \\ &= -\sigma \end{aligned}$$

which is the same as the RHS, since $j(\sigma) = 0$. If $\rho = \tau$ then

$$\begin{aligned}(\partial_m \circ P_{m-1} + P_{m-2} \circ \partial_{m-1})(\tau) &= \partial_m P_{m-1}(\tau) \\ &= -\partial_m \sigma \\ &= -\tau - \alpha,\end{aligned}$$

which is the same as the RHS, since $j(\tau) = -\alpha$.

Thus $i_* \circ j$ is chain homotopic to the identity. Thus i_* is a chain equivalence.