

FIRST MIDTERM
MATH 190B, UCSD, SPRING 25

You have 50 minutes.

There are 5 problems, and the total number of points is 55. Show all your work. *Please make your work as clear and easy to follow as possible.*

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Name:_____

Signature:_____

Student ID #:_____

Problem	Points	Score
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total	55	

1. (15pts) *Give the definition of a simplex.*

A simplex $\sigma \subset \mathbb{R}^n$ is the convex hull of $m \leq n + 1$ points in linear general position.

(ii) *Give the definition of a simplicial complex.*

A finite set $\mathcal{K}$ of simplices such that if

- $\sigma \in \mathcal{K}$ and τ is a face of σ then $\tau \in \mathcal{K}$, and if
- σ and $\tau \in \mathcal{K}$ and $\rho = \sigma \cap \tau$ is non-empty then ρ is a face of both σ and τ .

(iii) *Give the definition of a boundary.*

An m -chain α is a boundary if there is an $(m + 1)$ -chain β such that

$$\partial_{m+1}\beta = \alpha.$$

2. (10pts) *Let $\mathcal{K}$ be a simplicial complex. Show that the k -skeleton $\mathcal{L}$ is a simplicial complex.*

Suppose that $\sigma \in \mathcal{L}$ and τ is a face of σ . As $\sigma \in \mathcal{L}$, it follows that $\sigma \in \mathcal{K}$ and that σ has dimension at most k . As τ is a face of σ and $\mathcal{K}$ is a simplicial complex it follows that $\tau \in \mathcal{K}$. As τ is a face of σ it follows that τ has dimension at most k . But then $\tau \in \mathcal{L}$.

Now suppose that σ and $\tau \in \mathcal{L}$. Suppose $\rho = \sigma \cap \tau$ is non-empty. Then σ and $\tau \in \mathcal{K}$ and so ρ is a face of both σ and τ .

Hence $\mathcal{L}$ is a simplicial complex.

3. (10pts) *Show that any two m -simplices are homeomorphic.*

Let

$$\sigma = \langle v_0, v_1, \dots, v_m \rangle \subset \mathbb{R}^n.$$

be an m -simplex. It suffices to show that σ is homeomorphic to the standard m -simplex Δ^m inside $\mathbb{R}^{m+1}$, with vertices the standard basis vectors $e_0, e_1, \dots, e_m$ of $\mathbb{R}^{m+1}$.

I claim that every point p of σ is uniquely of the form

$$p = \sum \lambda_i v_i \quad \text{where} \quad \lambda_i \in [0, 1] \quad \text{and} \quad \sum \lambda_i = 1.$$

Let C be the set of all points of this form. Suppose that p and $q \in C$ and $t \in [0, 1]$. Then

$$\begin{aligned} tp + (1-t)q &= t \sum \lambda_i v_i + (1-t) \sum \mu_i v_i \\ &= \sum \nu_i v_i, \end{aligned}$$

where

$$v_i = t\lambda_i + (1-t)\mu_i \in [0, 1].$$

We have

$$\begin{aligned} \sum \nu_i &= \sum t\lambda_i + (1-t) \sum \mu_i \\ &= t \sum \lambda_i + (1-t) \sum \mu_i \\ &= 1. \end{aligned}$$

Thus C is convex. As C contains $v_0, v_1, \dots, v_m$ it follows that $\sigma \subset C$. Now suppose that $p \in \sigma$. Let

$$\mu_i = \frac{\lambda_i}{t} \quad \text{where} \quad t = \sum_{i < m} \lambda_i = 1 - \lambda_m.$$

Then $\sum \mu_i = 1$ and so $\mu_i \in [0, 1]$. Let $q = \sum \mu_i v_i$. Then $q \in \sigma$ by induction and $p = tq + (1-t)p_m \in \sigma$. Thus $C = \sigma$.

Suppose that $\sum \lambda_i v_i = \sum \mu_i v_i$. Then $\sum t_i u_i = 0$, where $u_i = v_i - v_0$ and $t_i = \lambda_i - \mu_i$. As the vectors $u_1, u_2, \dots, u_m$ are independent it follows that $t_i = 0$ so that $\lambda_i = \mu_i$. But then the expression is unique.

Consider the linear map

$$\mathbb{R}^{m+1} \longrightarrow \mathbb{R}^n$$

that sends e_i to v_i . This map is continuous as it is linear. By what we just proved this is a bijection. As Δ^n is compact and σ is Hausdorff it is a homeomorphism.

4. (10pts) Write down a triangulation $\mathcal{K}$ of an annulus and compute its homology.

Take the unit square and divide it into three congruent rectangles using the lines $x = 1/3$ and $x = 2/3$. Now divide each rectangle into two triangles by adding a diagonal to get six congruent triangles.

This gives a triangulation $\mathcal{K}$ of the unit square I^2 . Identifying the sides belonging to the lines $x = 0$ and $x = 1$ gives a triangulation of the annulus.

The annulus is connected and so

$$H_0(\mathcal{K}) \simeq \mathbb{Z}$$

with generator a vertex.

Let α be a 2-cycle. Pick a 2-simplex σ . Then one edge e of σ does not belong to any other 2-simplex. The coefficient of e in $\partial_2 \sigma$ is non-zero. Thus the coefficient of σ in α is zero. It follows that $\alpha = 0$.

Hence

$$H_2(\mathcal{K}) = 0.$$

Now consider the first homology. The condition that a 1-chain α is a 1-cycle is that it can be rearranged so that

$$\alpha = \sum_i \langle v_i, v_{i+1} \rangle,$$

where the first and last vertex are the same. Any such is a sum of cycles where there are no repetitions. In this example, considering the 1-skeleton, every cycle is a sum of three cycles, so that every third vertex is repeated.

Putting back the 2-simplices, kills all the cycles around a triangle. The only thing that remains are horizontal cycles. Let α be the one along the top. Then all other 1-cycles are homologous to a multiple of α .

If β is a 2-chain such that

$$m\alpha = \partial_2 \beta$$

then consider a 2-simplex σ at the bottom. If e is the edge along the bottom to which σ belongs, then e appears with coefficient ± 1 in $\partial_2 \sigma$. As e has coefficient zero in $m\alpha$ it follows that the coefficient of σ in β is zero. But then the three diagonal edges are exposed and repeating this argument with those edges, it follows that β must be zero.

Thus no multiple of α is homologous to zero. Therefore

$$H_1(\mathcal{K}) \simeq \mathbb{Z}$$

with generator α .

5. (10pts) *Let $\mathcal{K}$ be a simplicial complex. Show that the class of a single vertex is never zero in homology.*

Define a map

$$\delta: C_0(\mathcal{K}) \longrightarrow \mathbb{Z}.$$

by the rule

$$\alpha = \sum a_i v_i \longrightarrow \sum a_i$$

Note that δ is a group homomorphism. Suppose that β is a 1-chain,

$$\beta = \sum b_i \sigma_i.$$

Note that

$$\begin{aligned} \delta(\partial_1 \beta) &= \delta(\partial_1 \sum a_i \sigma_i) \\ &= \sum a_i \delta(\partial_1 \langle v_i, w_i \rangle) \\ &= \sum a_i \delta(v_i - w_i) \\ &= \sum a_i 0 \\ &= 0. \end{aligned}$$

Thus the boundaries belong to the kernel of δ . In other words if α is a chain homologous to zero then $\delta(\alpha) = 0$.

On the other hand,

$$\delta(v) = 1,$$

so that the class of a vertex is never homologous to zero.

Bonus Challenge Problems

6. (10pts) Write down a triangulation $\mathcal{K}$ of a Klein bottle and compute its homology.

We use a triangulation $\mathcal{K}$ of I^2 with the usual eighteen triangles. Note that the two vertical sides are identified with the opposite orientation. This means the projection map down to I is compatible with the identification of the sides. Let $\mathcal{L}$ be the simplicial complex you get by identifying the endpoints of I .

It follows that there is a simplicial map to $\mathcal{L}$.

The annulus is connected and so

$$H_0(\mathcal{K}) \simeq \mathbb{Z}$$

with generator a vertex.

Suppose that α is a 2-cycle. Let σ be a 2-simplex. If the coefficient of σ in α is zero then let e be an edge of σ . Arguing as in question 4, the 2-simplex that has e as an edge must also have coefficient zero. Continuing in this way, one can argue that $\alpha = 0$.

Suppose that α and β are two 2-cycles. If the coefficient of σ is a in α and b in β then $b\alpha - a\beta$ has coefficient zero. But then $b\alpha - a\beta = 0$, by what we already proved.

Thus every 2-cycle is a multiple of a single 2-cycle α . By symmetry the coefficients of α are ± 1 . Note that α is determined by the condition that every horizontal edge has coefficient zero in $\partial_2 \alpha$.

It is easy to check that then the coefficients of the two vertical edges along the sides don't cancel. Thus

$$H_2(\mathcal{K}) = 0.$$

We turn to computing the first homology. Arguing as in 4, every 1-cycle is a sum of a multiple of a horizontal 1-cycle α plus a multiple of a vertical 1-cycle β .

But the very fact that we cannot write down a 2-cycle means that there is a 2-cycle δ with coefficients 1, such that

$$\partial_2 \delta = 2\alpha - 2\beta.$$

Finally we know that

$$H_1(\mathcal{L}) \simeq \mathbb{Z}$$

generated by a 1-cycle γ . Thus pushforward of the horizontal 1-cycle α is equal to γ . Thus no multiple of α is homologous to zero. It follows that

$$H_1(\mathcal{L}) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$$

generated by α and the class of β modulo α .

7. (10pts) *Imagine a picture hanging from two nails in the wall. How do you loop the string around the two nails so that if you remove either nail the picture falls down?*

Here is a picture of a solution



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