

24. HOMOLOGY AND HOMOTOPY

The material from now on is not examinable. There are a couple of other quite general methods to compute the fundamental group. The first is van Kampen's theorem. Rather than state the result, which needs a little bit of background in group theory, let us give one example.

Example 24.1. *Let X be the union of two circles attached at one point x_0 .*

There are two obvious loops, a representing going around one circle and b going around the other. It is not hard to see that these two generate the fundamental group. Van Kampen's theorem says there are no relations between these two loops. Thus

$$\pi_1(X, x_0) = F_{a,b},$$

the free group on two generators.

This means that the universal cover is quite complicated. We already saw that the universal cover of the torus induces a cover of X . But the fundamental group of the torus is abelian, a small part of the fundamental group of X . Put differently the grid is far from simply connected (it is path connected).

In fact the situation is far worse than this.

Definition 24.2. *A **Cell complex** (or **CW complex**) is constructed iteratively*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X_n = X,$$

as follows. If have constructed X_{k-1} then we attach finitely many cells $e = e_\alpha$, each homeomorphic to the closed ball of dimension k , along a continuous map f of the boundary ∂e ,

$$f: \partial e \longrightarrow X_{k-1}.$$

Note that the boundary of a cell is a sphere. It is not hard to see that every triangulable space is homeomorphic to a CW complex. Many of the spaces we already saw have simple realisations as CW complexes. To get a sphere of dimension n , attach an n -cell to a point. Two circles attached at a point is a CW complex in an obvious way. We also saw a way to get the real projective plane as a CW complex.

Theorem 24.3. *Every finitely presented group is the fundamental group of a CW complex of dimension two.*

Sketch of Proof. Suppose we are given generators $a_1, a_2, \dots, a_n$ and relations $r_1, r_2, \dots, r_m$. First we construct a CW complex with free group on the generators. This is easy, take n circles attached at one point.

The generators $a_1, a_2, \dots, a_n$ correspond to loops around each circle. To impose the relations, take m two cells $e_1, e_2, \dots, e_m$. Writing out Each relation r_j as a word tells us how to divide the boundary of e_j into arcs and how to attach each arc to the boundary of one of the circles. $\square$

This seems like a nice result until you realise:

Theorem 24.4. *There is no algorithm to compute the fundamental group of a CW complex.*

Proof. There is no algorithm to decide if a finitely presented group is the trivial group. $\square$

There are some positive results.

Definition 24.5. *Let G be a group. The **commutator subgroup**, denoted $[G, G]$, is the group generated by the commutators*

$$[a, b] = a^{-1}b^{-1}ab.$$

The commutator subgroup $[G, G] \triangleleft G$ is normal in G .

The quotient group

$$A(G) = \frac{G}{[G, G]}$$

*is called the **abelianisation** of G , the largest abelian quotient of G . If $\phi: G \rightarrow H$ is a group homomorphism to any abelian group there is an induced group homomorphism $\psi: A(G) \rightarrow H$ which makes the following diagram commute*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \downarrow & \nearrow \psi & \\ A(G) & & \end{array}$$

Theorem 24.6 (Hurewicz Theorem). *Let X be a path connected space and let x_0 be a point of X .*

There is a natural group homomorphism

$$h_*: \pi_1(X, x_0) \rightarrow H_1(X)$$

which realises $H_1(X)$ as the abelianisation of the fundamental group.

Let $[\gamma] \in \pi_1(X, x_0)$ be an element of the fundamental group. As γ is a loop it is naturally a continuous function

$$\gamma: S^1 \rightarrow X.$$

S^1 is a simplicial complex and

$$H_1(S^1) \simeq \mathbb{Z}.$$

Pick a generator u and let

$$h_*([\gamma]) = \gamma_* u.$$

(24.6) is particularly useful if for some reason we know the fundamental group is abelian.

Theorem 24.7 (Poincaré Conjecture). *Let M be a 3-manifold.*

If

$$H_m(M) \simeq \begin{cases} \mathbb{Z} & \text{if } m = 0 \text{ or } 3 \\ 0 & \text{otherwise} \end{cases}$$

and M is simply connected then M is homeomorphic to S^3 .

(24.7) was proved by Pearlstein in 2004. In fact he proved much more. One can break a 3-manifold into pieces. He proved that the universal cover of each piece falls into one of eight cases with a particularly simple geometry.

There is a natural generalisation of the fundamental group.

Definition 24.8. *Let X be a topological space with base point x_0 . Then the set $\pi_n(X)$ of continuous maps, up to homotopy,*

$$f: S^n \longrightarrow X$$

sending p_0 to x_0 , is an abelian group for $n \geq 2$.

In fact one shows that we get an abelian group by using Homework 8, Question 6. π_n is a functor. There is a very beautiful and strong:

Theorem 24.9. *Let $f: X \longrightarrow Y$ be a continuous map.*

Then f is a homotopy equivalence if and only if $\pi_n(f): \pi_n(X) \longrightarrow \pi_n(Y)$ is an isomorphism.

Note that we need the map f . It is not enough to know that two topological spaces have the same homotopy groups.

Unfortunately it turns out to be fiendishly complicated to compute homotopy groups. We will probably never know $\pi_m(S^n)$ for all m and n .

One rather beautiful application of homotopy groups of spheres is to understanding normed algebras on $\mathbb{R}^n$. If $n = 1$ then we have the reals which is a field. If $n = 2$ then we get the complex numbers which is also a field. If $n = 4$ then we get the quaternions which is a division algebra (all the properties of a field, except drop the hypotheses that multiplication is commutative). If we further drop the hypotheses that

the multiplication is associative we get the octonions, which are a real vector space of dimension eight.

In fact one way to establish this is to use homotopy theory. Note that the units are a multiplicative subgroup and that they all have norm one. Thus the sphere S^{n-1} is a topological group. We already saw in the homework the curious connection between the fundamental group and topological groups.

There is an interesting generalisation of (24.7) to all dimensions:

Conjecture 24.10. *If M is a manifold (CW complex) homotopic to a sphere then it is homeomorphic to a sphere.*

This was proved in dimensions $n \geq 5$ by Smale. The case $n = 4$ was proved more recently by Friedman in 1982. In fact one can also consider the same question for smooth manifolds. In this case the conjecture is false. For example there are 27 exotic smooth 7-spheres,

$$X_k = \{ (a, b, c, d, e) \in \mathbb{C}^5 \mid a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0 \} \quad \text{for} \quad 1 \leq k \leq 28.$$

Theorem 24.11 (Hurewicz Theorem (bis)). *Let X be a simply connected space and let x_0 be a point of X .*

There are natural group homomorphisms

$$h_*: \pi_n(X, x_0) \longrightarrow H_n(X).$$

If $H_k(X) = 0$ for $k < n$ then

$$h_*: \pi_n(X, x_0) \longrightarrow H_n(X)$$

is an isomorphism.

This gives a reasonable way to compute the higher homotopy groups, provided we know the homology groups are zero.