

HOMEWORK 9, DUE WEDNESDAY JUNE 4TH

1. Let X be a topological space.

Show that X is simply connected if and only if for every pair of points p and q there is a unique homotopy class of path from p to q .

2. Let X and Y be topological spaces and let $p: X \times Y \rightarrow X$ denote projection onto the first factor.

If Y has the discrete topology then show that p is a covering map.

3. Let $q: X \rightarrow Y$ and $r: Y \rightarrow Z$ be covering maps.

If $r^{-1}(z)$ is finite, for every $z \in Z$, then show that $p = r \circ q: X \rightarrow Z$ is a covering map.

4. (i) Let

$$p: S^1 \rightarrow S^1$$

be the function

$$p(z) = z^2.$$

Show that p is a covering map.

(ii) What is the induced pushforward map on the fundamental group $\pi_1(S^1, 1)$?

(iii) More generally, if $p(z) = z^n$ then show that p is a covering map.

(iv) What is the induced pushforward map on the fundamental group $\pi_1(S^1, 1)$?

5. Compute the fundamental groups of the following surfaces:

(i) the annulus;

(ii) the torus;

(iii) the sphere;

(iv) the real projective plane.

Challenge Problems: (Just for fun)

6. Let G be a path connected, locally path connected (meaning that topological group and let $p: \tilde{G} \rightarrow G$ be a covering map, where $\tilde{G}$ is path connected. Pick a point e in the fibre over the identity e .

(i) Show that $\tilde{G}$ is a topological group, with identity e , and $\tilde{G}$ is unique such that p is a group homomorphism.

(ii) Show that the kernel of p lies in the centre of $\tilde{G}$.

(iii) Show that the matrix group $\text{SO}(3)$ of 3×3 real orthogonal matrices of determinant one (equivalently the group of rotations of $\mathbb{R}^3$) is homeomorphic to $\mathbb{RP}^3$, real projective three space.

(iv) Using (i) and (iii) gives S^3 the structure of a group. Identify the corresponding matrix group.